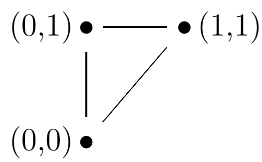
Math 362 **Exam 2 Solutions** **Fall '09**

1. Consider the density function

$$f(x,y) = \begin{cases} 15x^2y & \text{for } 0 \le x \le 1,\ x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) What is the marginal density function $f_2(y)$?

SOLUTION: The active region is



We have

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \int_0^y 15x^2y\,dx & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 5x^3y|_0^y = 5y^4 & \text{if } y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

b) What is $P(Y \ge \frac{1}{2})$?

SOLUTION:

$$\int_{\frac{1}{2}}^{\infty} f_2(y)\,dy = \int_{\frac{1}{2}}^{1} 5y^4\,dy = y^5|_{\frac{1}{2}}^{1} = \frac{31}{32}.$$

c) What is $P(X \le \frac{1}{4} \mid Y = \frac{1}{2})$?

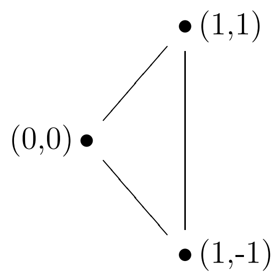
SOLUTION:

$$\int_0^{\frac{1}{4}} \frac{f(x,\frac{1}{2})}{f_2(\frac{1}{2})}\,dx = \int_0^{\frac{1}{4}} \frac{\frac{15}{2}x^2}{\frac{5}{16}}\,dx = 8x^3|_0^{\frac{1}{4}} = \frac{1}{8}.$$

2. Consider the density function

$$f(x,y)=\begin{cases}\frac{3}{2}(x+y) & \text{for } 0\le x\le 1,\ -x\le y\le x\\ 0 & \text{otherwise.}\end{cases}$$

SOLUTION: The active region is



a) What is the marginal density function $f_1(x)$?
SOLUTION:

$$f_1(x)=\begin{cases}\frac{3}{2}\int_{-x}^{x} x+y\,dy & \text{for } x\in[0,1]\\ 0 & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{3}{2}(xy+\frac{1}{2}y^2)|_{-x}^{x}=3x^2 & \text{for } x\in[0,1]\\ 0 & \text{otherwise.}\end{cases}$$

b) What is $P(X\le\frac{3}{4}, Y\le\frac{1}{2})$?
SOLUTION:

$$\int_0^{\frac{1}{2}}\int_{-x}^{x}\frac{3}{2}(x+y)\,dydx+\int_{\frac{1}{2}}^{\frac{3}{4}}\int_{-x}^{\frac{1}{2}}\frac{3}{2}(x+y)\,dydx$$

$$=\int_0^{\frac{1}{2}}\frac{3}{2}(xy+\tfrac{1}{2}y^2)|_{-x}^{x}\,dx+\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{3}{2}(xy+\tfrac{1}{2}y^2)|_{-x}^{\frac{1}{2}}\,dx$$

$$=\int_0^{\frac{1}{2}}3x^2\,dx+\int_{\frac{1}{2}}^{\frac{3}{4}}\frac{3}{4}x^2+\frac{3}{4}x+\frac{3}{16}\,dx$$

$$=\frac{1}{8}+\frac{61}{256}=\frac{93}{256}.$$

c) What is $P(X\le\frac{3}{4})$?
SOLUTION:

$$\int_0^{\frac{3}{4}}f_1(x)\,dx=\int_0^{\frac{3}{4}}3x^2\,dx=x^3|_0^{\frac{3}{4}}=\frac{27}{64}.$$

d) What is $P(Y\le\frac{1}{2}\mid X\le\frac{3}{4})$?
SOLUTION:

$$\frac{P(X\le\frac{3}{4},Y\le\frac{1}{2})}{P(X\le\frac{3}{4})}=\frac{\frac{93}{256}}{\frac{27}{64}}=\frac{31}{36}.$$

3. Let

$$f(y) = \begin{cases} ky^3 & \text{for } 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) For what value of k is this a density function?
SOLUTION: $\int_{-\infty}^{\infty} f(y)\,dy = \int_0^1 ky^3\,dy = \frac{k}{4}$, so $k = 4$.

b) For this value of k, what are the expected value and variance of Y?
SOLUTION:

$$E(Y) = \int_{-\infty}^{\infty} yf(y)\,dy = \int_0^1 4y^4\,dy = \frac{4}{5}y^5|_0^1 = \frac{4}{5}.$$

$$V(Y) = E(Y^2) - (E(Y))^2 = \int_{-\infty}^{\infty} y^2 f(y)\,dy - \frac{16}{25} = \int_0^1 4y^5\,dy - \frac{16}{25}$$
$$= \frac{2}{3} - \frac{16}{25} = \frac{2}{75}.$$