Math 362 Exam 1 Solutions Spring '11

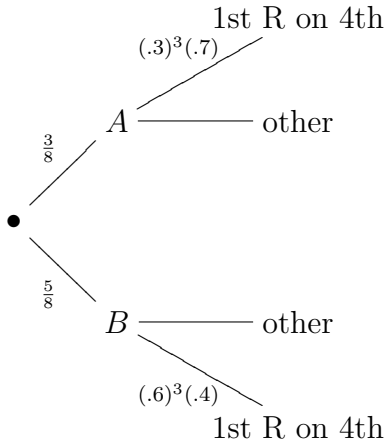
1. Urn A contains 7 red balls and 3 green balls. Urn B contains 4 red balls and 6 green balls. An experiment is conducted as follows:

   > Flip 3 coins. If you get exactly two heads, draw **with** replacement from Urn A until you get a red ball. Otherwise, draw **with** replacement from urn B until you get a red ball.

   a) What is the probability the first red ball comes on the fourth draw?
   b) What is the conditional probability you drew from Urn B given that the first red ball comes on the fourth draw?

   SOLUTION: If you flip 3 coins, the probability of two heads is given by the binomial distribution: $\binom{2}{3}(\frac{1}{2})^2(\frac{1}{2}) = \frac{3}{8}$. That gives the probabilities in the left-hand branch of the following tree diagram.

   Because we are drawing with replacement, the probabilities in the right-hand branches come from the geometric distributions for urns A and B, respectively. Here, when drawing from urn A, $p = .7$ and $q = .3$. When drawing from urn B, $p = .4$ and $q = .6$.



   For 1a), the probability is the sum of the probabilities of the branches leading to "1st R on 4th", so

$$P(\text{1st R on 4th}) = \frac{3}{8} \cdot (.3)^3(.7) + \frac{5}{8} \cdot (.6)^3(.4)$$

For 1b), we have

$$P(A \mid \text{1st R on 4th}) = \frac{P(A \cap \text{1st R on 4th})}{P(\text{1st R on 4th})} = \frac{\frac{3}{8} \cdot (.3)^3(.7)}{\frac{3}{8} \cdot (.3)^3(.7) + \frac{5}{8} \cdot (.6)^3(.4)}.$$

Here, the numerator is from the branch of the tree diagram going through both A and "1st R on 4th".

2. Roll a pair of dice. If the sum is less than 5 you win. Play this game six times.

   SOLUTION: Since you are playing a fixed number of times and the trials are independent, this is a binomial distribution with $n = 6$, and p is the probability of rolling a sum of less than 5 on a single roll. The ways you can roll a sum of less than 5 are

   $$(1,1),\ (1,2),\ (2,1),\ (1,3),\ (2,2),\ (3,1),$$

   so $p = \frac{6}{36} = \frac{1}{6}$. Write X for the number of successes in 6 rolls.

   a) What is the probability you lose more often than you win? SOLUTION: Here, we want

$$P(X=0) + P(X=1) + P(X=2) = \left(\frac{5}{6}\right)^6 + \binom{6}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 + \binom{6}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4.$$

   b) What is the expected number of wins? SOLUTION: The expected value is $np = 6 \cdot \frac{1}{6} = 1$.

   c) What is the variance for the number of wins? SOLUTION: The variance is $npq = 6 \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{6}$.

3. Deal a 7 card hand.

   SOLUTION: The drawing is without replacement and you are tracking the random variable X that records the number of aces in the hand. So this is a hypergeometric distribution with $n = 7$, $N = 52$, $r = 4$, so $p = \frac{4}{52} = \frac{1}{13}$ and $q = \frac{12}{13}$.

   a) What is probability you get less than 3 aces? SOLUTION:

$$P(X=0) + P(X=1) + P(X=2) = \frac{\binom{48}{7}}{\binom{52}{7}} + \frac{\binom{4}{1}\binom{48}{6}}{\binom{52}{7}} + \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}.$$

   b) How many aces do you expect there to be? SOLUTION: The expected value is $np = 7 \cdot \frac{1}{13} = \frac{7}{13}$.

c) What is the variance for this question? SOLUTION: The variance is

$$V(X) = npq \cdot \frac{N-n}{N-1} = 7 \cdot \frac{1}{13} \cdot \frac{12}{13} \cdot \frac{45}{51}.$$

4. Roll a die until you've gotten three sixes.

SOLUTION: This is a negative binomial distribution, X, with $p = \frac{1}{6}$ and the number of successes required being 3.

a) What is the probability you finish on the 7th roll? SOLUTION: $P(X = 7) = \binom{6}{2} p^3 q^{7-3} = \binom{6}{2} (\frac{1}{6})^3 (\frac{5}{6})^4$.

b) How many rolls do you expect to make? SOLUTION: $E(X) = \frac{3}{p} = 18$.

c) What is the variance for the number of rolls? SOLUTION: $V(X) = \frac{3q}{p^2} = 90$.

5. Deal 10 cards. What is the probability you get four of one kind and at least three of another kind?

SOLUTION: There are five different (and disjoint) ways you can get such a hand: (4,4,2), (4,4,1,1), (4,3,3), (4,3,2,1), (4,3,1,1,1).

Here, for instance, (4,3,1,1,1) means 4 of one kind, 3 of a second kind, and 1 each from a third, fourth and fifth kind. Here, a "kind" is one of A, K, Q, J or 2–9, so there are thirteen kinds, and there are four of each kind in the deck. The number of hands of type (4,3,1,1,1) is $\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}$, obtained by "pick a kind, then pick four of that kind; pick a second kind and then pic three of that kind; pick three additional kinds and pick one each of those three kinds." Thus,

$$P(4,3,1,1,1) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{10}}.$$

Similarly,

$$P(4,4,2) = \frac{\binom{13}{2}\binom{4}{4}\binom{4}{4}\binom{11}{1}\binom{4}{2}}{\binom{52}{10}},$$

$$P(4,4,1,1) = \frac{\binom{13}{2}\binom{4}{4}\binom{4}{4}\binom{11}{2}\binom{4}{1}\binom{4}{1}}{\binom{52}{10}},$$

$$P(4,3,3) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{2}\binom{4}{3}\binom{4}{3}}{\binom{52}{10}},$$

$$P(4,3,2,1) = \frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{3}\binom{11}{1}\binom{4}{2}\binom{10}{1}\binom{4}{1}}{\binom{52}{10}}.$$

Since the five different types of hand are disjoint, just add up the five displayed probabilities.

6. A particular locality experiences a major earthquake on an average of once every 3 years. Suppose the occurrences of major earthquakes in that locality follow a Poisson distribution.

   SOLUTION: This is a Poisson distribution with $\mu = \frac{1}{3}$, so $P(X = k) = \frac{(\frac{1}{3})^k}{k!} \cdot e^{-\frac{1}{3}}$.

   a) What is the probability they have two major quakes in 2015? SOLUTION: $P(X = 2) = \frac{1}{18} \cdot e^{-\frac{1}{3}}$.

   b) What is the probablilty they have more than one major quake in 2015? SOLUTION:

$$\begin{aligned} P(X > 1) &= 1 - (P(X = 0) + P(X = 1)) \\ &= 1 - \left(e^{-\frac{1}{3}} + \frac{1}{3} \cdot e^{-\frac{1}{3}} \right). \end{aligned}$$