Math 331 Extra problems: solutions Spring '08

1. Let ℓ be the line $x = 2$.

   a) Compute $\sigma_\ell \tau_{\left[\begin{smallmatrix} -2 \\ 2\sqrt{3} \end{smallmatrix}\right]}$ explicitly.

   SOLUTION: Let m be the line parallel to ℓ such that the directed distance from m to ℓ is $\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$. Thus, m is the line $x = 3$, and $\sigma_\ell \sigma_m = \tau_{\left[\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}\right]}$. Thus,

$$\begin{aligned}
\sigma_\ell \tau_{\left[\begin{smallmatrix} -2 \\ 2\sqrt{3} \end{smallmatrix}\right]} &= \sigma_\ell \tau_{\left[\begin{smallmatrix} -2 \\ 0 \end{smallmatrix}\right]} \tau_{\left[\begin{smallmatrix} 0 \\ 2\sqrt{3} \end{smallmatrix}\right]} \\
&= \sigma_\ell \sigma_\ell \sigma_m \tau_{\left[\begin{smallmatrix} 0 \\ 2\sqrt{3} \end{smallmatrix}\right]} \\
&= \sigma_m \tau_{\left[\begin{smallmatrix} 0 \\ 2\sqrt{3} \end{smallmatrix}\right]},
\end{aligned}$$

   a glide reflection with axis m, as $\left[\begin{smallmatrix} 0 \\ 2\sqrt{3} \end{smallmatrix}\right] \parallel m$.

   b) Compute $\tau_{\left[\begin{smallmatrix} 2 \\ -\frac{2}{\sqrt{3}} \end{smallmatrix}\right]} \sigma_\ell$ explicitly.

   SOLUTION: Let m be the line parallel to ℓ such that the directed distance from ℓ to m is $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Thus, m is the line $x = 3$, and $\sigma_m \sigma_\ell = \tau_{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]}$. Thus,

$$\begin{aligned}
\tau_{\left[\begin{smallmatrix} 2 \\ -2\sqrt{3} \end{smallmatrix}\right]} \sigma_\ell &= \tau_{\left[\begin{smallmatrix} 0 \\ -2\sqrt{3} \end{smallmatrix}\right]} \tau_{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]} \sigma_\ell \\
&= \tau_{\left[\begin{smallmatrix} 0 \\ -2\sqrt{3} \end{smallmatrix}\right]} \sigma_m \sigma_\ell \sigma_\ell \\
&= \tau_{\left[\begin{smallmatrix} 0 \\ -2\sqrt{3} \end{smallmatrix}\right]} \sigma_m,
\end{aligned}$$

   a glide reflection with axis m, as $\tau_{\left[\begin{smallmatrix} 0 \\ -2\sqrt{3} \end{smallmatrix}\right]} \parallel m$.

2. Let ℓ be the line $y = x + 2$.

a) Compute $\sigma_\ell \rho_{0,\frac{\pi}{2}}$ explicitly.

SOLUTION: Let m be the line $y = x$ and let n be the line $y = 0$. Then the directed angle from n to m is $\frac{\pi}{4}$, so $\sigma_m \sigma_n = \rho_{0,\frac{\pi}{2}}$. Also, the directed distance from m to ℓ is $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$, as the perpendicular to m and ℓ through 0 is the line $y = -x$, which intersects ℓ at $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$. Thus,

$$\begin{aligned}\sigma_\ell \rho_{0,\frac{\pi}{2}} &= \sigma_\ell \sigma_m \sigma_n \\ &= \tau_{\left[\begin{smallmatrix}-2\\2\end{smallmatrix}\right]} \sigma_n \\ &= \tau_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]} \tau_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]} \sigma_n \\ &= \tau_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]} \sigma_p \sigma_n \sigma_n,\end{aligned}$$

where p is the line $y = 1$,

$$= \tau_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]} \sigma_p,$$

a glide reflection with axis p, as $\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right] \parallel p$.

b) Compute $\rho_{0,\frac{3\pi}{2}} \sigma_\ell$ explicitly.

SOLUTION: Let m be the line $y = x$ and let n be the line through the origin such that the directed angle from m to n is $\frac{3\pi}{4}$. Then $\sigma_n \sigma_m = \rho_{0,\frac{3\pi}{2}}$. Note that n is the x-axis, $y = 0$. Note that the directed distance from ℓ to m is $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$, as the perpendicular to ℓ through the origin is the line $y = -x$, which meets ℓ at $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$. Thus, $\sigma_m \sigma_\ell = \tau_{\left[\begin{smallmatrix}2\\-2\end{smallmatrix}\right]}$. So

$$\begin{aligned}\rho_{0,\frac{3\pi}{2}} \sigma_\ell &= \sigma_n \sigma_m \sigma_\ell \\ &= \sigma_n \tau_{\left[\begin{smallmatrix}2\\-2\end{smallmatrix}\right]} \\ &= \sigma_n \tau_{\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]} \tau_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]} \\ &= \sigma_n \sigma_n \sigma_p \tau_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]},\end{aligned}$$

where p is the line $y = 1$

$$= \sigma_p \tau_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]},$$

a glide reflection with axis p, as $p \parallel \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$.