Math 331 Exam 1 Solutions  Spring 2014

1. Let $\ell$ be the line $y = -\frac{1}{\sqrt{3}} x + \frac{2}{\sqrt{3}}$. Let $\alpha = \sigma_{\ell} \rho_{(0, \frac{\pi}{3})}$. Write $\alpha$ in standard form (i.e., as a translation, rotation, reflection, or glide reflection in standard form).

**Solution:** we write $\rho_{(0, \frac{\pi}{3})} = \sigma_{m} \sigma_{n}$ with $m \parallel \ell$. Thus,

$$\text{slope}(m) = \text{slope}(\ell) = -\frac{1}{\sqrt{3}}.$$ 

Since $m \cap n = 0$, $m$ is the line $y = -\frac{1}{\sqrt{3}} x$. The directed angle from $n$ to $m$ is $\frac{\pi}{2} \cdot \frac{\pi}{3}$, so the directed angle from $m$ to $n$ is $-\frac{\pi}{6}$. Since the directed angle from the positive $x$-axis to $m$ is $-\frac{\pi}{6}$, the slope of $n$ is $\tan^{-1}(-\frac{\pi}{3}) = -\sqrt{3}$. Since $0 \in n$, the equation for $n$ is $y = -\sqrt{3} x$. Thus,

$$\alpha = \sigma_{\ell} \sigma_{m} \sigma_{n} = \tau_{v} \sigma_{n},$$

where $v$ is twice the directed distance from $m$ to $\ell$. To find that distance, we let $q$ be the line through the origin perpendicular to $\ell$ and $m$. So

$$\text{slope}(q) = -\frac{1}{\text{slope}(m)} = \sqrt{3}.$$ 

So $q$ is the line $y = \frac{1}{\sqrt{3}} x$. Now $v = 2(\ell \cap q - m \cap q) = 2(\ell \cap q)$. To calculate $\ell \cap q$ we set

$$\sqrt{3} x = -\frac{1}{\sqrt{3}} x + \frac{2}{\sqrt{3}}$$

$$3x = -x + 2$$

$$x = \frac{1}{2}.$$ 

So $y = \frac{\sqrt{3}}{2}$, and $v = 2\left[\begin{array}{c} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{array}\right] = \left[\begin{array}{c} 1 \\ \sqrt{3} \end{array}\right]$. So

$$\alpha = \tau_{\left[\begin{array}{c} 1 \\ \sqrt{3} \end{array}\right]} \sigma_{m}.$$ 

To put this in standard form we need to write $v = w + z$ with $w \parallel n$ and $z \perp n$. Now, $n = \text{span}(u)$, where $u$ is the unit vector $\left[\begin{array}{c} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{array}\right]$. Then $u^\perp = \left[\begin{array}{c} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{array}\right]$. Since $u, u^\perp$ is an orthonormal basis of $\mathbb{R}^2$, we
have

\[ v = \langle v, u \rangle u + \langle v, u \perp \rangle u \perp \]

\[ = \left[ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} \right] \left[ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right] + \left[ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right] \]

\[ = -\left[ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right] + \sqrt{3} \left[ \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} \right] \]

\[ = \left[ \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{3}}{4} \end{bmatrix} \right] + \left[ \begin{bmatrix} \frac{3}{\sqrt{3}} \\ \frac{\sqrt{3}}{4} \end{bmatrix} \right] \]

\[ = w + z. \]

Note it is easy to check that \( w + z = v \), providing a check on your work.

Now \( \tau_v = \tau_w \tau_z \), so

\[ \alpha = \tau_w (\tau_z \sigma_n) \]

\[ = \tau_w \sigma_{\tau_z (n)}, \]

since \( z \perp n \). Since \( w \parallel n \parallel \tau_z (n) \), this is a glide reflection in standard form, and it suffices to calculate \( \tau_z (n) \). Since \( 0 \in n, \frac{z}{2} \in \tau_z (n) \). Note

\[ \frac{z}{2} = \left[ \begin{bmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \end{bmatrix} \right] \]

and since slope(\( n \)) = \(-\sqrt{3}\), the point-slope formula for \( \tau_z (n) \) is

\[
\frac{y - \frac{\sqrt{3}}{4}}{x - \frac{3}{4}} = -\sqrt{3} \\
\frac{y - \sqrt{3}}{4} = -\sqrt{3} x + \frac{3\sqrt{3}}{4} \\
y = -\sqrt{3} x + \sqrt{3}
\]
2. Let \( \alpha = \rho\left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \frac{\pi}{3}\right)\rho(0, \frac{\pi}{3}) \). Write \( \alpha \) in standard form (i.e., as a translation, rotation, reflection, or glide reflection in standard form).

**Solution:** Write \( \rho\left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \frac{\pi}{3}\right) = \sigma_L \sigma_m \) and write \( \rho(0, \frac{\pi}{3}) = \sigma_m \sigma_n \).

Then,

\[
\ell \cap m = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \quad \text{and the directed angle from } m \text{ to } \ell \text{ is } \frac{\pi}{6},
\]

\[
m \cap n = 0 \quad \text{and the directed angle from } n \text{ to } m \text{ is } \frac{\pi}{6}.
\]

Since both 0 and \( \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \) lie on \( m \), \( m = \text{span} \left( \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \right) \), which is the line \( y = \sqrt{3}x \). From the directed angle, we see \( \ell \) is vertical. Since \( \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \in \ell \), \( \ell \) is the line \( x = 1 \). Finally, the directed angles show that \( n \) has slope \( \frac{1}{\sqrt{3}} \). Since \( 0 \in n \), \( n \) is the line \( y = \frac{1}{\sqrt{3}}x \). We have

\[
\alpha = \sigma_L \sigma_m \sigma_n = \sigma_L \sigma_n = \rho(\ell \cap n, \frac{\pi}{3}) = \rho\left(\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}, \frac{\pi}{3}\right).
\]