Math 331 Exam 1 Solutions Spring '11

1. Let ℓ be the line $y = -\sqrt{3}\,x + 2$. Compute $\Omega_\ell \rho_{0,-\frac{\pi}{3}}$ explicitly.

   SOLUTION: Write $\rho_{0,-\frac{\pi}{3}} = \Omega_m \Omega_n$ with $m \| \ell$. Thus,

   $$\text{slope}(m) = \text{slope}(\ell) = -\sqrt{3}.$$

   Since $m \cap n = 0$, m is the line $y = -\sqrt{3}\,x$. Since $\rho_{0,-\frac{\pi}{3}} = \Omega_m \Omega_n$, the directed angle from n to m is $-\frac{\pi}{6}$, so the directed angle from m to n is $\frac{\pi}{6}$.

   Since the directed angle from the positive x-axis to m is $-\frac{\pi}{3}$, the directed angle from the x-axis to n is $-\frac{\pi}{6}$. Thus slope$(n) = -\frac{1}{\sqrt{3}}$. Since $0 \in n$, n is the line $y = -\frac{1}{\sqrt{3}}\,x$. We have

   $$\Omega_\ell \rho_{0,-\frac{\pi}{3}} = \Omega_\ell \Omega_m \Omega_n.$$

   Since $m \| \ell$,

   $$\Omega_\ell \Omega_m = \tau_{2v},$$

   where v is the directed distance from m to ℓ, i.e., if q is a line perpendicular to ℓ and m,

   $$v = q \cap \ell - q \cap m.$$

   We have

   $$\text{slope}(q) = -\frac{1}{\text{slope}(m)} = \frac{1}{\sqrt{3}}.$$

   We may take $0 \in q$, in which case q is the line $y = \frac{1}{\sqrt{3}}\,x$. Since $0 \in m$, $0 = q \cap m$, so $v = q \cap \ell$, which we compute by setting

   $$\frac{1}{\sqrt{3}}\,x = -\sqrt{3}\,x + 2.$$

   Multiplying both sides by $\sqrt{3}$, we get

   $$\begin{aligned} x &= -3x + 2\sqrt{3} \\ 4x &= 2\sqrt{3} \\ x &= \frac{\sqrt{3}}{2}. \end{aligned}$$

   So $y = \frac{1}{\sqrt{3}}\,x = \frac{1}{2}$, hence $v = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, so

   $$\Omega_\ell \Omega_m = \tau_{(\sqrt{3},1)}, \quad \text{hence}$$
   $$\Omega_\ell \rho_{0,-\frac{\pi}{3}} = \tau_{(\sqrt{3},1)} \Omega_n.$$

To proceed, we write $(\sqrt{3}, 1) = w + z$ with $w \| n$ and $z \perp n$. To do this, note that since the directed angle from the positive x-axis to n is $-\frac{\pi}{6}$ and since $0 \in n$, $n = [u]$ for

$$u = \left(\cos\left(-\frac{\pi}{6}\right), \sin\left(-\frac{\pi}{6}\right)\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

Since u is a unit vector, a unit normal to n is given by

$$N = u^{\perp} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

By construction, u, N is an orthonormal basis of $\mathbb{R}^2$, so

$$\begin{aligned}(\sqrt{3}, 1) &= \langle(\sqrt{3}, 1), u\rangle u + \langle(\sqrt{3}, 1), N\rangle N \\ &= u + \sqrt{3}\, N \\ &= \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right).\end{aligned}$$

So set $w = u = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ and $z = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$. We have

$$\begin{aligned}\tau_{(\sqrt{3},1)}\Omega_n &= \tau_w \tau_z \Omega_n \\ &= \tau_w \Omega_{n+\frac{1}{2}z},\end{aligned}$$

as $z \perp n$. Since $w \| n$, this is a glide reflection in standard form, and it suffices to compute the line $p = n + \frac{1}{2}z$. Since $0 \in n$, $\frac{1}{2}z \in p$. So the point-slope form of p is

$$\frac{y - \frac{3}{4}}{x - \frac{\sqrt{3}}{4}} = -\frac{1}{\sqrt{3}},$$

giving

$$y = -\frac{1}{\sqrt{3}}\, x + 1.$$

2. Compute $\rho_{(2,0),\frac{\pi}{3}} \rho_{0,\frac{2\pi}{3}}$ explicitly.

SOLUTION: We write $\rho_{(2,0),\frac{\pi}{3}} = \Omega_\ell \Omega_m$ and $\rho_{0,\frac{2\pi}{3}} = \Omega_m \Omega_n$, so

$$\begin{aligned}&\ell \cap m = (2, 0), && \text{the directed angle from } m \text{ to } \ell \text{ is } \frac{\pi}{6}, \\ &m \cap n = 0, && \text{the directed angle from } n \text{ to } m \text{ is } \frac{\pi}{3}.\end{aligned}$$

Since m goes through both 0 and $(2, 0)$ it is the x-axis, i.e., the line $y = 0$. Since the directed angle from m to n is $-\frac{\pi}{3}$, the slope of n is $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$. Since $0 \in n$, n is the line $y = -\sqrt{3}\, x$.

Since the directed angle from m to ℓ is $\frac{\pi}{6}$, ℓ has slope $\frac{1}{\sqrt{3}}$. Since $(2,0) \in \ell$, the point-slope formula for ℓ is

$$\frac{y-0}{x-2} = \frac{1}{\sqrt{3}},$$

so ℓ is the line $y = \frac{1}{\sqrt{3}}\,x - \frac{2}{\sqrt{3}}$.

Since the sum of the two angles of rotation is π,

$$\rho_{(2,0),\frac{\pi}{3}}\rho_{0,\frac{2\pi}{3}} = \rho_{P,\pi},$$

where $P = \ell \cap n$. Thus, P is obtained by solving

$$\frac{1}{\sqrt{3}}\,x - \frac{2}{\sqrt{3}} = -\sqrt{3}\,x, \quad \text{so}$$
$$x - 2 = -3x$$

Thus, $x = \frac{1}{2}$, hence $y = -\frac{\sqrt{3}}{2}$, so $P = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

3. Compute $\rho_{(1,\sqrt{3}),-\pi}\rho_{0,\pi}$ explicitly.

SOLUTION: Here, the sum of the rotational angles is 0, so the composite is a translation. As above, we write $\rho_{(1,\sqrt{3}),-\pi} = \Omega_\ell\Omega_m$ and $\rho_{0,\pi} = \Omega_m\Omega_n$, so that

$$\ell \cap m = (1,\sqrt{3}), \qquad \text{the directed angle from } m \text{ to } \ell \text{ is } -\frac{\pi}{2},$$
$$m \cap n = 0, \qquad \text{the directed angle from } n \text{ to } m \text{ is } \frac{\pi}{2}.$$

Thus, m is the line through 0 and $(1,\sqrt{3})$, i.e., m is the line $y = \sqrt{3}\,x$. So the angle from the positive x-axis to m is $\arctan\sqrt{3} = \frac{\pi}{3}$.

By the directed angles given above, ℓ and n both have slope $-\frac{1}{\sqrt{3}}$ (in fact, both are perpendicular to m as the angles are $\pm\frac{\pi}{2}$ — but the angles, and hence the slope, can be computed directly without that).

Since $0 \in n$, n is the line $y = -\frac{1}{\sqrt{3}}\,x$. We use the point-slope formula for ℓ:

$$\frac{y-\sqrt{3}}{x-1} = -\frac{1}{\sqrt{3}}$$
$$y - \sqrt{3} = -\frac{1}{\sqrt{3}}\,x + \frac{1}{\sqrt{3}}$$
$$y = -\frac{1}{\sqrt{3}}\,x + \frac{4}{\sqrt{3}}.$$

Since $\ell \| n$,

$$\rho_{(1,\sqrt{3}),-\pi}\rho_{0,\pi} = \Omega_\ell \Omega_n = \tau_{2v},$$

where v is the directed distance from n to ℓ. To compute v, note that m is, in fact, perpendicular to ℓ and n, so that

$$v = \ell \cap m - n \cap m = (1, \sqrt{3}) - 0 = (1, \sqrt{3}),$$

so

$$\rho_{(1,\sqrt{3}),-\pi}\rho_{0,\pi} = \tau_{(2,2\sqrt{3})}.$$

Note that we actually didn't need to calculate the slope-intercept form of ℓ.