Math 331 Exam 1 Solutions Spring '10

1. Let ℓ be the line $y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$. Compute $\rho_{0,\frac{\pi}{3}}\Omega_\ell$ explicitly. SOLUTION: We write
$$\rho_{0,\frac{\pi}{3}}\Omega_\ell = \Omega_m\Omega_n\Omega_\ell,$$
with $n||\ell$, $m \cap n = 0$, and the directed angle from $n \to m$ equal to $\frac{\pi}{6}$. Thus, n is the line $y = -\frac{1}{\sqrt{3}}x$ and m is the x-axis.

   Since $n||\ell$, $\Omega_n\Omega_\ell$ is the translation by twice the directed distance from ℓ to n. To find the directed distance, let q be the perpendicular to n through the origin. Thus q is the line $y = \sqrt{3}x$. The direct distance, then, is $-P$, where $P = \ell \cap q$. Setting $\sqrt{3}x = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$ and solving, we obtain $P = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, hence $\Omega_n\Omega_\ell = \tau_{(-1,-\sqrt{3})}$. We obtain
$$\begin{aligned}\rho_{0,\frac{\pi}{3}}\Omega_\ell = \Omega_m\Omega_n\Omega_\ell &= \Omega_m\tau_{(-1,-\sqrt{3})} \\ &= \Omega_m\tau_{(0,-\sqrt{3})}\tau_{(-1,0)} \\ &= \Omega_m\Omega_m\Omega_p\tau(-1,0) = \Omega_p\tau(-1,0).\end{aligned}$$
Here, we use the fact that $(0, -\sqrt{3}) \perp m$, so that $\tau_{(0,-\sqrt{3})} = \Omega_m\Omega_p$, where the directed distance from p to m is $(0, -\frac{\sqrt{3}}{2})$. So p is the line $y = \frac{\sqrt{3}}{2}$ and the result is a glide reflection with axis p.

2. Let ℓ be the line $y = -\frac{1}{\sqrt{3}}x + \frac{2}{\sqrt{3}}$. Write $(0,1)$ as the sum of a vector parallel to ℓ and a vector perpendicular to ℓ.

   SOLUTION: ℓ is the line $(0, \frac{2}{\sqrt{3}}) + [u]$, where u is the unit vector $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. Thus, a unit normal for ℓ is given by $N = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. We obtain
$$\begin{aligned}(0,1) &= \langle(0,1), u\rangle u + \langle(0,1), N\rangle N \\ &= -\frac{1}{2}u + \frac{\sqrt{3}}{2}N = \left(-\frac{\sqrt{3}}{4}, \frac{1}{4}\right) + \left(\frac{\sqrt{3}}{4}, \frac{3}{4}\right)\end{aligned}$$

3. Compute $\rho_{0,\frac{\pi}{2}}\rho_{(2,2),\frac{\pi}{2}}$ explicitly. SOLUTION: we write
$$\rho_{0,\frac{\pi}{2}}\rho_{(2,2),\frac{\pi}{2}} = \Omega_\ell\Omega_m\Omega_m\Omega_n,$$
where m goes through both 0 and $(2,2)$, ℓ goes through 0 such that the directed angle from m to ℓ is $\frac{\pi}{4}$, and n goes through $(2,2)$ such that the directed angle from n to m is $\frac{\pi}{4}$. Thus, m is the line $y = x$, ℓ is the y-axis, and n is the line $y = 2$. We get
$$\rho_{0,\frac{\pi}{2}}\rho_{(2,2),\frac{\pi}{2}} = \Omega_\ell\Omega_n = \rho_{(0,2),\pi}.$$

4. Compute $\rho_{(2,0),\frac{\pi}{3}}\rho_{0,-\frac{\pi}{3}}$ explicitly. SOLUTION: we write
$$\rho_{(2,0),\frac{\pi}{3}}\rho_{0,-\frac{\pi}{3}} = \Omega_\ell\Omega_m\Omega_m\Omega_n,$$
where m goes through both $(2,0)$ and 0, ℓ goes through $(2,0)$ such that the directed angle from m to ℓ is $\frac{\pi}{6}$, and n goes through 0 such that the directed angle from n to m is $-\frac{\pi}{6}$.

Thus, m is the x-axis and n is the line $y = \frac{1}{\sqrt{3}}x$, and ℓ goes through $(2,0)$ with slope $\frac{1}{\sqrt{3}}$. So ℓ is the line $y = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$. We have
$$\rho_{(2,0),\frac{\pi}{3}}\rho_{0,-\frac{\pi}{3}} = \Omega_\ell\Omega_n.$$
Since n and ℓ are parallel, the result is the translation by twice the directed distance from n to ℓ. To find the directed distance, we let q be the perpendicular to n through 0. Thus, q is the line $y = -\sqrt{3}\,x$. We get
$$\rho_{(2,0),\frac{\pi}{3}}\rho_{0,-\frac{\pi}{3}} = \tau_{2P},$$
where $P = \ell \cap q = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Thus,
$$\rho_{(2,0),\frac{\pi}{3}}\rho_{0,-\frac{\pi}{3}} = \tau_{(1,-\sqrt{3})}.$$

5. Compute $\tau_{(2,-2)}\rho_{0,\frac{\pi}{2}}$ explicitly. SOLUTION: we write
$$\tau_{(2,-2)}\rho_{0,\frac{\pi}{2}} = \Omega_\ell\Omega_m\Omega_m\Omega_n,$$
where m is perpendicular to $(2,-2)$ and goes through 0, n goes through 0 such that the directed angle from n to m is $\frac{\pi}{4}$, and $\ell||m$ such that the directed distance from m to ℓ is $(1,-1)$.

Thus, m is the line $y = x$, n is the x-axis, and ℓ is the line through $(1,-1)$ with slope 1, which gives $y = x - 2$. We get
$$\tau_{(2,-2)}\rho_{0,\frac{\pi}{2}} = \Omega_\ell\Omega_n = \rho_{P,\frac{\pi}{2}},$$
where $P = \ell \cap n = (2,0)$.