Math 331 **Exam 1 solutions** **Spring '08**

1. Compute $\rho_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right],\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right],\frac{\pi}{2}}$ explicitly.

   SOLUTION: We write $\rho_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right],\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right],\frac{\pi}{2}} = \sigma_\ell\sigma_m\sigma_m\sigma_n$. This says ℓ and m go through $\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]$ with the directed angle from m to ℓ equal to $\frac{\pi}{4}$, while m and n go through $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ with the directed angle from n to m $\frac{\pi}{4}$. Thus, m goes through both $\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$, and hence has slope 1 and y-intercept $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$, so m is the line $y = x + 2$.

   Since the directed angle from m to ℓ is $\frac{\pi}{4}$, ℓ is vertical. Since ℓ goes through $\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]$, ℓ is the line $x = -2$.

   Since the directed angle from n to m is $\frac{\pi}{4}$, n is horizontal. Since n goes through $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$, n is the line $y = 2$. Thus,

$$\begin{aligned}\rho_{\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right],\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right],\frac{\pi}{2}} &= \sigma_\ell\sigma_m\sigma_m\sigma_n\\ &= \sigma_\ell\sigma_n\\ &= \rho_{\ell\cap n,\pi}\\ &= \rho_{\left[\begin{smallmatrix}-2\\2\end{smallmatrix}\right],\pi}.\end{aligned}$$

2. Compute $\rho_{\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right],\frac{2\pi}{3}}\rho_{0,-\frac{2\pi}{3}}$ explicitly.

   SOLUTION: We write $\rho_{\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right],\frac{2\pi}{3}}\rho_{0,-\frac{2\pi}{3}} = \sigma_\ell\sigma_m\sigma_m\sigma_n$. This says ℓ and m go through $\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]$ with the directed angle from m to ℓ equal to $\frac{\pi}{3}$, while m and n go through 0 with the directed angle from n to m equal to $-\frac{\pi}{3}$. Thus m goes through both $\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]$ and 0, hence m is the y-axis, $x = 0$.

   Since the directed angle from m to ℓ is $\frac{\pi}{3}$, ℓ has slope $-\frac{1}{\sqrt{3}}$. The point-slope formula for ℓ is

$$\frac{y-(-2)}{x-0} = -\frac{1}{\sqrt{3}},$$

   so the equation for ℓ is $y = -\frac{1}{\sqrt{3}}x - 2$.

   Since the directed angle from n to m is $-\frac{\pi}{3}$, n also has slope $-\frac{1}{\sqrt{3}}$. Since n goes through the origin, the equation for n is $y = -\frac{1}{\sqrt{3}}x$.

   As above, $\rho_{\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right],\frac{2\pi}{3}}\rho_{0,-\frac{2\pi}{3}} = \sigma_\ell\sigma_n$. Since ℓ and n are parallel, this gives a translation. To find the translation vector, we must first find a perpendicular, q, to both ℓ and n, which, for convenience may as well go through the origin.

The slope of q is $-\frac{1}{\text{slope}\,\ell} = \sqrt{3}$, so q is the line $y = \sqrt{3}x$. The directed distance from n to ℓ is $n \cap q - \ell \cap q = 0 - \ell \cap q = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{3}{2} \end{bmatrix}$. So the translation vector is twice that directed distance, giving

$$\rho_{\left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right], \frac{2\pi}{3}} \rho_{0, -\frac{2\pi}{3}} = \tau_{\left[\begin{smallmatrix} -\sqrt{3} \\ -3 \end{smallmatrix}\right]}$$

3. Compute $\rho_{0,\pi}\tau_{\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]}$ explicitly.

SOLUTION: We write $\rho_{0,\pi}\tau_{\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]} = \sigma_\ell \sigma_m \sigma_m \sigma_n$. This says ℓ and m go through 0 with the directed angle from m to ℓ equal to $\frac{\pi}{2}$, while m and n are both perpendicular to $\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]$ with the directed distance from n to m equal to $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$. Thus, m goes through 0 and is perpendicular to $\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]$, and hence has slope 1. Thus m is the line $y = x$.

Since ℓ goes through 0 and the directed angle from m to ℓ is $\frac{\pi}{2}$, ℓ is the line $y = -x$.

Since $n \parallel m$ and the directed distance from n to m is $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$, n goes through $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. The slope of n is 1, so the point-slope formula for n is

$$\frac{y+1}{x-1} = 1,$$

so the equation fo n is $y = x - 2$. An easy calculation shows that $\ell \cap n = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, so

$$\rho_{0,\pi}\tau_{\left[\begin{smallmatrix} -2 \\ 2 \end{smallmatrix}\right]} = \rho_{\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], \pi}.$$

4. Let ℓ be the line $y = 0$ (the x-axis), m the line $y = \frac{1}{\sqrt{3}}x$, and n the line $y = \sqrt{3}x$. Compute $\sigma_n \sigma_m \sigma_\ell$ explicitly.

SOLUTION: The directed angle from ℓ to m is $\frac{\pi}{6}$. Since both go through 0, $\sigma_m \sigma_\ell = \rho_{0,\frac{\pi}{3}}$. Thus,

$$\begin{aligned} \sigma_n \sigma_m \sigma_\ell &= \sigma_n \rho_{0,\frac{\pi}{3}} \\ &= \sigma_n \sigma_n \sigma_m, \end{aligned}$$

since the directed angle from m to n is $\frac{\pi}{6}$,

$$= \sigma_m.$$

5. Let ℓ be the line $y = -x - 2$. Compute $\sigma_\ell \rho_{0,\frac{3\pi}{2}}$ explicitly.

SOLUTION: Let m be the line parallel to ℓ through 0. Then m is the line $y = -x$. Let n be the line through 0 such that the directed angle from n to m is $\frac{3\pi}{4}$. Then n is the x-axis, and $\sigma_m \sigma_n = \rho_{0,\frac{3\pi}{2}}$.

We next calculate the directed distance from m to ℓ. To do this, note that the line, q, through 0 perpendicular to m and ℓ is $y = x$. q intersects ℓ at $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$. So the directed distance from m to ℓ is $\begin{bmatrix} -1 \\ -1 \end{bmatrix} - 0 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. So $\sigma_\ell \sigma_m$ is the translation by twice this distance, or $\tau_{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}$. Thus,

$$\begin{aligned}
\sigma_\ell \rho_{0,\frac{3\pi}{2}} &= \sigma_\ell \sigma_m \sigma_n \\
&= \tau_{\begin{bmatrix} -2 \\ -2 \end{bmatrix}} \sigma_n \\
&= \tau_{\begin{bmatrix} -2 \\ 0 \end{bmatrix}} \tau_{\begin{bmatrix} 0 \\ -2 \end{bmatrix}} \sigma_n \\
&= \tau_{\begin{bmatrix} -2 \\ 0 \end{bmatrix}} \sigma_r \sigma_n \sigma_n,
\end{aligned}$$

where r is the line $y = -1$,

$$= \tau_{\begin{bmatrix} -2 \\ 0 \end{bmatrix}} \sigma_r,$$

a glide reflection with axis r, as $\begin{bmatrix} -2 \\ 0 \end{bmatrix} \parallel r$.