Math 331 Exam 1 Solutions Spring '06

1. Compute $\rho_{O,\frac{2\pi}{3}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\pi}$ explicitly.

   SOLUTION: $\rho_{O,\frac{2\pi}{3}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\pi} = \sigma_\ell\sigma_m\sigma_m\sigma_n = \sigma_\ell\sigma_n$, where m goes through O and $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, hence m is the x-axis.

   ℓ goes throurh O, and the directed angle from m to ℓ is $\frac{1}{2}\cdot\frac{2\pi}{3} = \frac{\pi}{3}$, so ℓ is the line $y = \sqrt{3}\,x$.

   n goes through $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, and the directed angle from n to m is $\frac{1}{2}\cdot\pi$, so n is vertical: n is the line $x = 2$.

   It remains to compute $\sigma_\ell\sigma_n$. By a theorem in class, the composite of two rotations is a rotation by the sum of the rotational angles, i.e., by $\frac{2\pi}{3} + \pi = \frac{5\pi}{3}$, unless that sum is a multiple of 2π (which in this case, it is not). Also, since ℓ and n are not parallel, $\sigma_\ell\sigma_n$ is a rotation about $\ell \cap n$. Thus,

   $$\rho_{O,\frac{2\pi}{3}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\pi} = \rho_{P,\frac{5\pi}{3}}, \quad \text{where } P = \ell \cap n.$$

   It remains to find P. Since n is the line $x = 2$ and ℓ is given by $y = \sqrt{3}\,x$, $P = \left[\begin{smallmatrix}2\\2\sqrt{3}\end{smallmatrix}\right]$.

2. Compute $\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],-\frac{\pi}{2}}$ explicitly.

   SOLUTION: $\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],-\frac{\pi}{2}} = \sigma_\ell\sigma_m\sigma_m\sigma_n = \sigma_\ell\sigma_n$, where m goes through O and $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, hence m is the x-axis.

   ℓ goes through O, and the directed angle from m to ℓ is $\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$, so ℓ is the line $y = x$.

   n goes through $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, and the directed angle from n to m is $\frac{1}{2}\cdot -\frac{\pi}{2} = -\frac{\pi}{4}$, so the directed angle from m to n is $\frac{\pi}{4}$. Thus n has slope 1. Since n goes through $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$, n is the line $y = x - 2$.

   Since ℓ and n are parallel, $\sigma_\ell\sigma_n$ is the translation by twice the directed distance from n to ℓ. To find the directed distance, we let q by any line perpendicular to n and ℓ, and let $Q = \ell \cap q$ and $P = n \cap q$. We then get

   $$\sigma_\ell\sigma_n = \tau_{O,2(Q-P)}.$$

   To find Q and P, we set q to be the line $y = -x$. This is perpendicular to ℓ and n, as required. Then $Q = O$, and $P = \left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$. This now gives

   $$\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],-\frac{\pi}{2}} = \tau_{O,\left[\begin{smallmatrix}-2\\2\end{smallmatrix}\right]}.$$

3. Compute $\tau_{O,\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]}\rho_{O,\frac{\pi}{2}}$ explicitly.

SOLUTION: $\tau_{O,\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]}\rho_{O,\frac{\pi}{2}} = \sigma_\ell\sigma_m\sigma_m\sigma_n = \sigma_\ell\sigma_n$, where $m \perp \overline{O\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]}$ and goes through O. Thus m is the line $y = -x$.

$\ell \parallel m$, and the directed distance from m to ℓ is $\frac{1}{2}\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$. Thus, $\ell = m + \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, the line through $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ with slope 1. Thus, ℓ is the line $y = -x + 2$.

n goes through O, and the directed angle from n to m is $\frac{1}{2}\cdot\frac{\pi}{2}$. Thus, n is the y-axis: $x = 0$.

By a theorem in class, $\tau_{O,\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]}\rho_{O,\frac{\pi}{2}}$ is a rotation by the same angle as that in $\rho_{O,\frac{\pi}{2}}$. And the rotation must be about the point $P = \ell \cap n = \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$. Thus,

$$\tau_{O,\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]}\rho_{O,\frac{\pi}{2}} = \rho_{\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right],\frac{\pi}{2}}.$$

4. Let ℓ be the line $y = 0$ (the x-axis), and m the line $y = \frac{1}{\sqrt{3}}x$. Compute $\sigma_m\sigma_\ell\sigma_m$ explicitly.

SOLUTION:

$$\sigma_m\sigma_\ell\sigma_m = \rho_{O,\theta}\sigma_m,$$

where θ is twice the directed angle from ℓ to m

$$= \rho_{O,\frac{\pi}{3}}\sigma_m$$
$$= \sigma_q\sigma_m\sigma_m = \sigma_q,$$

where the directed angle from m to q is $\frac{1}{2}\cdot\frac{\pi}{3}$, so q is the line $y = \sqrt{3}\,x$.

5. Let ℓ be the line $y = 2x + 3$, m the line $y = 2x - 1$, and n the line $y = -\frac{1}{2}x$. Calculate the following explicitly:
   a) $\sigma_n\sigma_\ell$. SOLUTION: n and ℓ are perpendicular, so $\sigma_n\sigma_\ell = \rho_{P,\pi}$, where $P = n \cap \ell = \begin{bmatrix} -\frac{6}{5} \\ \frac{3}{5} \end{bmatrix}$.
   b) $\sigma_n\sigma_m$. SOLUTION: n and m are perpendicular, so $\sigma_n\sigma_m = \rho_{Q,\pi}$, where $Q = n \cap m = \begin{bmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{bmatrix}$.
   c) $\sigma_m\sigma_\ell$. SOLUTION: $m \parallel \ell$, so $\sigma_m\sigma_\ell$ is the translation by twice the directed distance from ℓ to m. Since n is perpendicular to m and ℓ, the directed distance is $Q - P$, with P and Q as above. Thus,

$$\sigma_m\sigma_\ell = \tau_{O,\begin{bmatrix} \frac{16}{5} \\ -\frac{8}{5} \end{bmatrix}}.$$