1. Compute $\rho_{[0]}^{[1]} \circ \rho_{[0]}^{[1]} : \pi$ explicitly.

**Solution:** Write $\ell$ for the line $y = -x + 1$, $m$ for the line $x = 1$ and $n$ for the line $y = 1$:

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3. Let $\ell$ be the line $y = 2x + 5$, $m$ the line $y = 2x + 1$, and $n$ the line $y = -\frac{1}{2}x$. Calculate the following explicitly:

a) $\sigma_n\sigma_\ell$

**Solution:** $\ell \perp n$, so $\sigma_n\sigma_\ell = \rho_{P,\pi}$, where $P = \ell \cap n$. To find $P$, set $2x + 5 = -\frac{1}{2}x$. We get $x = \frac{2}{5}(-5) = -2$. So $y = -\frac{1}{2}(-2) = 1$, hence $P = [-2, 1]^T$.

b) $\sigma_n\sigma_m$

**Solution:** $m \perp n$, so $\sigma_n\sigma_m = \rho_{Q,\pi}$, where $Q = m \cap n$. To find $Q$, set $2x + 1 = -\frac{1}{2}x$. We get $x = \frac{2}{5}(-1) = -\frac{2}{5}$. So $y = -\frac{1}{2}(-\frac{2}{5}) = \frac{1}{5}$, hence $Q = [-\frac{2}{5}, \frac{1}{5}]^T$.

c) $\sigma_m\sigma_\ell$

**Solution:** $m \parallel \ell$, so $\sigma_m\sigma_\ell$ is the translation by twice the directed distance from $\ell$ to $m$. Since $n$ is perpendicular to $\ell$ and $m$, the directed distance from $\ell$ to $m$ is $m \cap n - \ell \cap n$, or $Q - P$, with $P$ and $Q$ as above. So $\sigma_m\sigma_\ell = \tau_{O,2(Q-P)} = \tau_{O,\frac{10}{3}[\frac{1}{3}, \frac{2}{5}]^T}$.

4. Let $\ell$ be the line $y = 0$ (the $x$-axis), $m$ the line $x = 0$ (the $y$-axis), $n$ the line $y = \sqrt{3}x$. Compute the following explicitly:

a) $\sigma_n\sigma_\ell\sigma_m$

**Solution:** $\ell \cap n = O$, and the directed angle from $\ell$ to $n$ is $\frac{\pi}{3}$, so $\sigma_n\sigma_\ell = \rho_{O,\frac{\pi}{3}} = \sigma_q\sigma_m$, where $q$ goes through $O$ and makes an angle of $\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$ with respect to the $x$-axis. So the equation of $q$ is $y = -\frac{1}{\sqrt{3}}x$. We have $\sigma_n\sigma_\ell\sigma_m = \rho_{O,\frac{2\pi}{3}}\sigma_m = \sigma_q\sigma_m\sigma_m = \sigma_q$.

b) $\sigma_n\sigma_\ell\sigma_n$

**Solution:** As above, $\sigma_n\sigma_\ell = \rho_{O,\frac{2\pi}{3}}$. Here, we write $\rho_{O,\frac{2\pi}{3}} = \sigma_r\sigma_n$, where the directed angle from $n$ to $r$ is $\frac{\pi}{3}$. So $r$ makes an angle of $\frac{2\pi}{3}$ with respect to the $x$-axis, hence $r$ has the formula $y = -\sqrt{3}x$. We have $\sigma_n\sigma_\ell\sigma_n = \sigma_r\sigma_n\sigma_n = \sigma_r$. 

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**Exam 1 Solutions**

3. Let $\ell$ be the line $y = 2x + 5$, $m$ the line $y = 2x + 1$, and $n$ the line $y = -\frac{1}{2}x$. Calculate the following explicitly:

a) $\sigma_n\sigma_\ell$

**Solution:** $\ell \perp n$, so $\sigma_n\sigma_\ell = \rho_{P,\pi}$, where $P = \ell \cap n$. To find $P$, set $2x + 5 = -\frac{1}{2}x$. We get $x = \frac{2}{5}(-5) = -2$. So $y = -\frac{1}{2}(-2) = 1$, hence $P = [-2, 1]^T$.

b) $\sigma_n\sigma_m$

**Solution:** $m \perp n$, so $\sigma_n\sigma_m = \rho_{Q,\pi}$, where $Q = m \cap n$. To find $Q$, set $2x + 1 = -\frac{1}{2}x$. We get $x = \frac{2}{5}(-1) = -\frac{2}{5}$. So $y = -\frac{1}{2}(-\frac{2}{5}) = \frac{1}{5}$, hence $Q = [-\frac{2}{5}, \frac{1}{5}]^T$.

c) $\sigma_m\sigma_\ell$

**Solution:** $m \parallel \ell$, so $\sigma_m\sigma_\ell$ is the translation by twice the directed distance from $\ell$ to $m$. Since $n$ is perpendicular to $\ell$ and $m$, the directed distance from $\ell$ to $m$ is $m \cap n - \ell \cap n$, or $Q - P$, with $P$ and $Q$ as above. So $\sigma_m\sigma_\ell = \tau_{O,2(Q-P)} = \tau_{O,\frac{10}{3}[\frac{1}{3}, \frac{2}{5}]^T}$.

4. Let $\ell$ be the line $y = 0$ (the $x$-axis), $m$ the line $x = 0$ (the $y$-axis), $n$ the line $y = \sqrt{3}x$. Compute the following explicitly:

a) $\sigma_n\sigma_\ell\sigma_m$

**Solution:** $\ell \cap n = O$, and the directed angle from $\ell$ to $n$ is $\frac{\pi}{3}$, so $\sigma_n\sigma_\ell = \rho_{O,\frac{\pi}{3}} = \sigma_q\sigma_m$, where $q$ goes through $O$ and makes an angle of $\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$ with respect to the $x$-axis. So the equation of $q$ is $y = -\frac{1}{\sqrt{3}}x$. We have $\sigma_n\sigma_\ell\sigma_m = \rho_{O,\frac{2\pi}{3}}\sigma_m = \sigma_q\sigma_m\sigma_m = \sigma_q$.

b) $\sigma_n\sigma_\ell\sigma_n$

**Solution:** As above, $\sigma_n\sigma_\ell = \rho_{O,\frac{2\pi}{3}}$. Here, we write $\rho_{O,\frac{2\pi}{3}} = \sigma_r\sigma_n$, where the directed angle from $n$ to $r$ is $\frac{\pi}{3}$. So $r$ makes an angle of $\frac{2\pi}{3}$ with respect to the $x$-axis, hence $r$ has the formula $y = -\sqrt{3}x$. We have $\sigma_n\sigma_\ell\sigma_n = \sigma_r\sigma_n\sigma_n = \sigma_r$. 

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