Math 331  Exam 1 Solutions  Spring ’05

1. Compute $\rho_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \pi}^{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]}$ explicitly.

   **Solution:** Let $\ell$ be the line $y = 0$, $m$ the line through $\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$ with slope $-1$ and $n$ the line through the origin with slope 1. Then $\ell \cap m = \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$ and the angle from $m$ to $\ell$ is $\frac{\pi}{4}$, so $\rho_{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right], \pi}^{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]} = \sigma_\ell \sigma_m$.

   Note that the point slope formula for $m$ gives
   
   $$\frac{y - 0}{x - 2} = -1,$$

   so $m$ is the line $y = -x + 2$.

   The line $n$, of course, is $y = x$. Since $\ell \cap n = O$ and the angle from $\ell$ to $n$ is $\frac{\pi}{4}$, we have $\rho_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \pi}^{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]} = \sigma_n \sigma_\ell$. Thus,

   $$\rho_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \pi}^{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]} = \sigma_n \sigma_\ell \sigma_m = \sigma_n \sigma_m.$$

   Now, the point of intersection of $m$ and $n$ is given by solving $-x + 2 = x$, hence $x = 1$, so $y = 1$, hence $m \cap n = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Since the angle from $m$ to $n$ is $\frac{\pi}{2}$, we get

   $$\rho_{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \pi}^{\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]} = \rho_{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \pi}.$$ 

2. Compute $\rho_{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \frac{\pi}{2}}^{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]}$ explicitly.

   **Solution:** Here, we let $\ell$ be the line $y = -x + 1$, which goes through both $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. We choose $m$ to be the line through $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ such that the angle from $m$ to $\ell$ is $\frac{\pi}{4}$, as this will give

   $$\rho_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \frac{\pi}{2}}^{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]} = \sigma_\ell \sigma_m.$$

   From the angles given, we see that $m$ must be vertical, so $m$ is the line $x = 1$.

   Now let $n$ be the line through $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ such that the angle from $\ell$ to $n$ is $-\frac{\pi}{4}$. This will give

   $$\rho_{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], -\frac{\pi}{2}}^{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} = \sigma_n \sigma_\ell.$$

   Again, $n$ must be vertical, so $n$ is the line $x = 0$.

   Now

   $$\rho_{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], -\frac{\pi}{2}}^{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \frac{\pi}{2}} = \sigma_n \sigma_\ell \sigma_m = \sigma_n \sigma_m.$$
Since \( m \) and \( n \) are parallel, \( \sigma_n \sigma_m \) is the translation by twice the directed distance from \( m \) to \( n \), so
\[
\rho_{[0]} \cdot \frac{\rho_{[1]} \cdot \rho_{[0]}}{2} = \tau_{O, \frac{\rho_{[1]} \cdot \rho_{[0]}}{2}}.
\]

3. Compute \( \tau_{O, \frac{0}{2}} \rho_{O, \frac{z_3}{2}} \) explicitly.

**SOLUTION:** As usual, we write \( \tau_{O, \frac{0}{2}} \rho_{O, \frac{z_3}{2}} = \sigma_n \sigma_\ell \) and \( \rho_{O, \frac{z_3}{2}} = \sigma_\ell \sigma_m \).

The latter forces \( \ell \) to go through the origin, while the former forces \( \ell \) to be perpendicular to \( \overline{O \cdot \frac{0}{2}} \); in fact, the directed distance from \( \ell \) to \( n \) must be \( \frac{1}{2} [0 \cdot 2] = [0 \cdot 1] \). Thus, \( \ell \) is the \( x \)-axis, and \( n \) is the line \( y = -1 \).

Now \( \rho_{O, \frac{z_3}{2}} = \sigma_\ell \sigma_m \) forces \( m \) to go through the origin, such that the angle from \( m \) to \( \ell \) is \( \frac{\pi}{3} \). Since \( \ell \) is the \( x \)-axis this forces \( m \) to have slope \( -\sqrt{3} \). So \( m \) is the line \( y = -\sqrt{3} x \).

As before, \( \tau_{O, \frac{0}{2}} \rho_{O, \frac{z_3}{2}} = \sigma_n \sigma_\ell \sigma_m \), the rotation around \( n \cap \ell \) by twice the angle from \( m \) to \( n \). Since \( n \parallel \ell \), the angle is \( \frac{\pi}{3} \). The point of intersection is found by setting \( -1 = -\sqrt{3} x \), hence \( x = \frac{1}{\sqrt{3}} \) and \( y = -1 \), so
\[
\tau_{O, \frac{0}{2}} \rho_{O, \frac{z_3}{2}} = \rho_{\left[ \frac{1}{\sqrt{3}} \right], \frac{2\pi}{3}}
\]

4. Let \( \ell \) be the line \( y = 0 \) (the \( x \)-axis), and \( m \) the line \( y = x \). Compute \( \sigma_m \sigma_\ell \sigma_m \) explicitly.

**SOLUTION:** The angle from \( \ell \) to \( m \) is \( \frac{\pi}{4} \), so
\[
\sigma_m \sigma_\ell = \rho_{O, \frac{\pi}{4}} = \sigma_q \sigma_m,
\]
where \( q \) goes through the origin and the angle from \( m \) to \( q \) is \( \frac{\pi}{4} \). Thus, \( q \) is the \( y \)-axis. We have
\[
\sigma_m \sigma_\ell \sigma_m = \sigma_q \sigma_m \sigma_m = \sigma_q.
\]

5. Let \( \ell \) be the line \( y = 4x + 6 \), \( m \) the line \( y = 4x - 2 \), and \( n \) the line \( y = -\frac{1}{4}x \). Calculate the following explicitly:

a) \( \sigma_n \sigma_\ell \)

**SOLUTION:** \( P = n \cap \ell \) is given by setting \( -\frac{1}{3} x = 4x + 6 \), or \( \frac{17}{4} x = -6 \). So \( x = -\frac{24}{17} \), hence \( y = \frac{6}{17} \). So \( P = \left[ \frac{-24}{17}, \frac{6}{17} \right] \). Since the angle from \( \ell \) to \( n \) is \( \frac{\pi}{2} \),
\[
\sigma_n \sigma_\ell = \rho_{\left[ \frac{-24}{17}, \frac{6}{17} \right], \frac{\pi}{2}}
\]
b) $\sigma_n \sigma_m$

Solution: $Q = n \cap m$ is given by setting $-\frac{1}{2}x = 4x - 2$, or $\frac{17}{4}x = 2$. So $x = \frac{8}{17}$, hence $y = \frac{2}{17}$. So $Q = \left[ \frac{8}{17} \right]$. Since the angle from $\ell$ to $n$ is $\frac{\pi}{2}$,

$$\sigma_n \sigma_\ell = \rho \left[ \frac{8}{17}, \frac{2}{17} \right], \pi$$

c) $\sigma_m \sigma_\ell$.

Solution: Since $n \parallel \ell$, $\sigma_m \sigma_\ell$ is translation by twice the directed distance from $\ell$ to $n$. Since $n$ is perpendicular to $\ell$ and $m$, that distance is $Q - P = \left[ \frac{\frac{17}{17}}{\frac{17}{17}} \right]$, so

$$\sigma_m \sigma_\ell = \tau \left[ \frac{64}{17}, \frac{16}{17} \right]$$