Math 331 Exam 1 Solutions Spring '05

1. Compute $\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\frac{\pi}{2}}$ explicitly.

   SOLUTION: Let ℓ be the line $y = 0$, m the line through $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ with slope -1 and n the line through the origin with slope 1. Then $\ell \cap m = \left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ and the angle from m to ℓ is $\frac{\pi}{4}$, so $\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \sigma_\ell\sigma_m$. Note that the point slope formula for m gives

   $$\frac{y-0}{x-2} = -1,$$

   so m is the line $y = -x + 2$.

   The line n, of course, is $y = x$. Since $\ell \cap n = O$ and the angle from ℓ to n is $\frac{\pi}{4}$, we have $\rho_{O,\frac{\pi}{2}} = \sigma_n\sigma_\ell$. Thus,

   $$\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \sigma_n\sigma_\ell\sigma_\ell\sigma_m = \sigma_n\sigma_m.$$

   Now, the point of intersection of m and n is given by solving $-x + 2 = x$, hence $x = 1$, so $y = 1$, hence $m \cap n = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. Since the angle from m to n is $\frac{\pi}{2}$, we get

   $$\rho_{O,\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \rho_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right],\pi}.$$

2. Compute $\rho_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],-\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\frac{\pi}{2}}$ explicitly.

   SOLUTION: Here, we let ℓ be the line $y = -x + 1$, which goes through both $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We choose m to be the line through $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ such that the angle from m to ℓ is $\frac{\pi}{4}$, as this will give

   $$\rho_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \sigma_\ell\sigma_m.$$

   From the angles given, we see that m must be vertical, so m is the line $x = 1$.

   Now let n be the line through $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ such that the angle from ℓ to n is $-\frac{\pi}{4}$. This will give

   $$\rho_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],-\frac{\pi}{2}} = \sigma_n\sigma_\ell.$$

   Again, n must be vertical, so n is the line $x = 0$.

   Now

   $$\rho_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],-\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \sigma_n\sigma_\ell\sigma_\ell\sigma_m = \sigma_n\sigma_m.$$

Since m and n are parallel, $\sigma_n\sigma_m$ is the translation by twice the directed distance from m to n, so

$$\rho_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right],-\frac{\pi}{2}}\rho_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],\frac{\pi}{2}} = \tau_{O,\left[\begin{smallmatrix}-2\\0\end{smallmatrix}\right]}.$$

3. Compute $\tau_{O,\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]}\rho_{O,\frac{2\pi}{3}}$ explicitly.

SOLUTION: As usual, we write $\tau_{O,\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]} = \sigma_n\sigma_\ell$ and $\rho_{O,\frac{2\pi}{3}} = \sigma_\ell\sigma_m$. The latter forces ℓ to go through the origin, while the former forces ℓ to be perpendicular to $\overline{O\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]}$: in fact, the directed distance from ℓ to n must be $\frac{1}{2}\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]$. Thus, ℓ is the x-axis, and n is the line $y = -1$.

Now $\rho_{O,\frac{2\pi}{3}} = \sigma_\ell\sigma_m$ forces m to go through the origin, such that the angle from m to ℓ is $\frac{\pi}{3}$. Since ℓ is the x-axis this forces m to have slope $-\sqrt{3}$. So m is the line $y = -\sqrt{3}\,x$.

As before, $\tau_{O,\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]}\rho_{O,\frac{2\pi}{3}} = \sigma_n\sigma_m$, the rotation around $n \cap m$ by twice the angle from m to n. Since $n \| \ell$, the angle is $\frac{\pi}{3}$. The point of intersection is found by setting $-1 = -\sqrt{3}\,x$, hence $x = \frac{1}{\sqrt{3}}$ and $y = -1$, so

$$\tau_{O,\left[\begin{smallmatrix}0\\-2\end{smallmatrix}\right]}\rho_{O,\frac{2\pi}{3}} = \rho_{\left[\begin{smallmatrix}\frac{1}{\sqrt{3}}\\-1\end{smallmatrix}\right],\frac{2\pi}{3}}$$

4. Let ℓ be the line $y = 0$ (the x-axis), and m the line $y = x$. Compute $\sigma_m\sigma_\ell\sigma_m$ explicitly.

SOLUTION: The angle from ℓ to m is $\frac{\pi}{4}$, so

$$\sigma_m\sigma_\ell = \rho_{O,\frac{\pi}{2}} = \sigma_q\sigma_m,$$

where q goes through the origin and the angle from m to q is $\frac{\pi}{4}$. Thus, q is the y-axis. We have

$$\sigma_m\sigma_\ell\sigma_m = \sigma_q\sigma_m\sigma_m = \sigma_q.$$

5. Let ℓ be the line $y = 4x + 6$, m the line $y = 4x - 2$, and n the line $y = -\frac{1}{4}x$. Calculate the following explicitly:

a) $\sigma_n\sigma_\ell$

SOLUTION: $P = n \cap \ell$ is given by setting $-\frac{1}{4}x = 4x + 6$, or $\frac{17}{4}x = -6$. So $x = \frac{-24}{17}$, hence $y = \frac{6}{17}$. So $P = \begin{bmatrix}\frac{-24}{17}\\ \frac{6}{17}\end{bmatrix}$. Since the angle from ℓ to n is $\frac{\pi}{2}$,

$$\sigma_n\sigma_\ell = \rho_{\left[\begin{smallmatrix}\frac{-24}{17}\\ \frac{6}{17}\end{smallmatrix}\right],\pi}$$

b) $\sigma_n\sigma_m$

SOLUTION: $Q = n \cap m$ is given by setting $-\frac{1}{4}x = 4x - 2$, or $\frac{17}{4}x = 2$. So $x = \frac{8}{17}$, hence $y = \frac{-2}{17}$. So $Q = \begin{bmatrix} \frac{8}{17} \\ \frac{-2}{17} \end{bmatrix}$. Since the angle from ℓ to n is $\frac{\pi}{2}$,

$$\sigma_n\sigma_\ell = \rho_{\begin{bmatrix} \frac{8}{17} \\ \frac{-2}{17} \end{bmatrix},\pi}$$

c) $\sigma_m\sigma_\ell$.

SOLUTION: Since $n\|\ell$, $\sigma_m\sigma_\ell$ is translation by twice the directed distance from ℓ to n. Since n is perpendicular to ℓ and m, that distance is $Q - P = \begin{bmatrix} \frac{32}{17} \\ \frac{-8}{17} \end{bmatrix}$, so

$$\sigma_m\sigma_\ell = \tau_{O,\begin{bmatrix} \frac{64}{17} \\ \frac{-16}{17} \end{bmatrix}}$$