1. Let
\[ f = (x^2 - 1)^3(x^2 + 1)^4(x - 1)^6(x - 2)^7 \quad \text{and} \]
\[ g = (x^2 - 1)^5(x^2 + 1)^2(x - 1)^4(x - 3)^2. \]
   a) What is the g.c.d. of \( f \) and \( g \) in \( \mathbb{Q}[X] \)?
   b) What is the least common multiple of \( f \) and \( g \) in \( \mathbb{Q}[X] \)?

2. Let \( f = x^3 + 2x^2 + 4 \). In which of the following is \( f \) irreducible?
   Show your work.
   a) \( \mathbb{Q}[x] \)  b) \( \mathbb{Z}_3[x] \)  c) \( \mathbb{Z}_5[x] \)  d) \( \mathbb{Z}_7[x] \)  e) \( \mathbb{R}[x] \)

3. Let \( f = 5x^6 + 2x^4 + 15x^2 - 12 \). List all the candidates for rational roots of \( f \), as identified by the theorem on rational roots.

4. Determine whether the following polynomial is irreducible in \( \mathbb{Q}[x] \) or not. **Show your work.** If you use Eisenstein’s criterion, say which prime you used and what verifications you made to show the criterion is satisfied.
   \[ f = 2x^4 + 30x^3 + 60x^2 + 90 \]

5. Give prime factorizations in \( \mathbb{Z}_2[x] \) for the following polynomials. Give proofs that the factors are prime. (You may use the results from class on the irreducibles in degrees \( \leq 2 \).
   a) \( f = x^6 + x^5 + x^3 + x + 1 \)
   b) \( f = x^6 + x^4 + x^3 + x^2 + x + 1 \)

6. Let
\[ f = (x^2 - 1)^2(x^2 + 1)^5(x - 1)^4(x - 2)^2 \quad \text{and} \]
\[ g = (x^2 - 1)^4(x^2 + 1)^3(x - 1)^3(x - 3)^3. \]
   a) What is the g.c.d. of \( f \) and \( g \) in \( \mathbb{Q}[X] \)?
   b) What is the least common multiple of \( f \) and \( g \) in \( \mathbb{Q}[X] \)?

7. Let \( f = x^3 + 7x + 29 \). In which of the following is \( f \) irreducible?
   Show your work.
   a) \( \mathbb{Q}[x] \)  b) \( \mathbb{Z}_2[x] \)  c) \( \mathbb{Z}_3[x] \)  d) \( \mathbb{Z}_5[x] \)  e) \( \mathbb{Z}_7[x] \)

8. Let \( f = 7x^5 + 3x^4 + 15x^2 + 10 \). List all the candidates for rational roots of \( f \), as identified by the theorem on rational roots.
9. Determine whether the following polynomial is irreducible in $\mathbb{Q}[x]$ or not. **Show your work.** If you use Eisenstein’s criterion, say which prime you used and what verifications you made to show the criterion is satisfied.

$$f = 3x^4 + 150x^3 + 30x^2 + 60$$

10. Give prime factorizations in $\mathbb{Z}_2[x]$ for the following polynomials. Give proofs that the factors are prime. (You may use the results from class on the irreducibles in degrees $\leq 2$.

a) $f = x^6 + x^5 + x^3 + x + 1$

b) $f = x^6 + x^4 + x^3 + x^2 + x + 1$

11. Let $f = 5x^7 + 29x^3 - 14$. List all the candidates for rational roots of $f$, as identified by the theorem on rational roots.

12. Determine whether the following polynomials are irreducible in $\mathbb{Q}[X]$ or not. **Show your work.** If you use Eisenstein’s criterion, say which prime you used.

a) $f = 10X^7 + 60X^5 + 90X^2 + 15$

b) $f = X^6 + 2X^5 + 2X + 4$

c) $f = X^8 + 30X^5 + 12X^2 + 18$

13. a) Show that $f = X^3 + X + 3$ is irreducible in $\mathbb{Q}[X]$.

b) Let $F = \mathbb{Q}[X]/(f)$ and let $\alpha = [X]_f$. What is the inverse of $\alpha + 1$ in $F^\times$?

14. Let

$$f = (x^2 + x + 1)^2(x + 1)^3(x - \frac{2}{3})(x - 2)^9$$

$$g = (x^2 + x + 1)^5(x + 1)(x - \frac{2}{3})^3(x - 5)^6.$$ 

a) What is the g.c.d. of $f$ and $g$ in $\mathbb{Q}[X]$?

b) What is the least common multiple of $f$ and $g$ in $\mathbb{Q}[X]$?

15. Give prime factorizations of the following elements of $\mathbb{Z}_2[x]$. Prove that each of the factors is irreducible. (We showed in class which polynomials of degree $\leq 2$ were irreducible. You may use this in your work.)

a) $f = x^6 + x^5 + x^4 + x^3 + 1$

b) $f = x^6 + x^5 + x^2 + 1$

c) $f = x^6 + x^5 + x^2 + 1$. 