Math 326  Exam 3  Spring ’05

1. Let \( f = 3x^5 + 2x^4 + 8x^2 + 10 \). List all the candidates for rational roots of \( f \), as identified by the theorem on rational roots.

2. Give a prime factorization in \( \mathbb{Z}_2[x] \) for the following polynomial. Give proofs that the factors are prime. (You may use the results from class on the irreducibles in degrees \( \leq 4 \).)

\[
f = x^7 + x^3 + x + 1
\]

3. Let \( f = x^3 + 4x + 2 \). In which of the following is \( f \) irreducible?
   a) \( \mathbb{Q}[x] \)
   b) \( \mathbb{Z}_2[x] \)
   c) \( \mathbb{Z}_3[x] \)
   d) \( \mathbb{Z}_5[x] \)
   e) \( \mathbb{R}[x] \)

4. Let \( f = x^4 + x^3 + x^2 + x + 1 \in \mathbb{Z}_2[x] \), and let \( \mathbb{F} = \mathbb{Z}_2[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements of \( \mathbb{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbb{F}^\times \)?
   c) What is the order of \( \alpha + 1 \) in \( \mathbb{F}^\times \)?
   d) Find a primitive element of \( \mathbb{F} \).

5. Let \( f = x^2 - 2 \in \mathbb{Z}_{11}[x] \), and let \( \mathbb{F} = \mathbb{Z}_{11}[x]/(f) \), a field. Let \( \alpha = [x]_f \).
   a) What are the possible orders of the elements of \( \mathbb{F}^\times \)?
   b) What is the order of \( \alpha \) in \( \mathbb{F}^\times \)?
   c) What is the order of \( \alpha^{72} \) in \( \mathbb{F}^\times \)?
   d) What is the order of \( 7\alpha \) in \( \mathbb{F}^\times \)?
   e) What is the order of \( \alpha + 1 \) in \( \mathbb{F}^\times \)?