Show all of your work.

1. Prove that

\[ 1 + \cdots + n = \frac{n(n + 1)}{2} \]

for all \( n \geq 1 \).

**SOLUTION:** We argue by induction on \( n \). The \( n \)-th statement, \( P_n \), is that \( 1 + \cdots + n = \frac{n(n+1)}{2} \).

\( P_1 \) just says \( 1 = \frac{2}{2} \), which is true. So let \( n \geq 1 \), and suppose that \( P_n \) is true. It suffices to show this implies that \( P_{n+1} \) is true, i.e., that

\[ 1 + \cdots + (n+1) = \frac{(n+1)(n+2)}{2}. \]

Now,

\[ 1 + \cdots + (n+1) = (1 + \cdots + n) + (n+1) \]
\[ = \frac{n(n+1)}{2} + (n+1) \]

by the inductive assumption

\[ = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \]
\[ = \frac{n(n+1) + 2(n+1)}{2} \]
\[ = \frac{(n+1)(n+2)}{2}. \]

The last statement follows by factoring \( n+1 \) out of \( n(n+1) + 2(n+1) \) (i.e., by the distributive law), and is what we needed to show.
2. Use the Euclidean Algorithm to find \((4097, 1751)\), and find a solution to Bezout’s identity.

**Solution:** The Euclidean Algorithm gives
\[
\begin{align*}
4097 &= 2 \cdot 1751 + 595 \\
1751 &= 2 \cdot 595 + 561 \\
595 &= 561 + 34 \\
561 &= 16 \cdot 34 + 17 \\
34 &= 2 \cdot 17,
\end{align*}
\]
so \((4097, 1751) = 17\). Now we back-substitute to solve Bezout’s identity.

\[
17 = 561 - 16 \cdot 34 = 561 - 16(595 - 561)
\]
\[
= -16(595) + 17(561) = -16(595) + 17(1751 - 2 \cdot 595)
\]
\[
= 17(1751) - 50(595) = 17(1751) - 50(4097 - 2 \cdot 1751)
\]
\[
= -50(4097) + 117(1751).
\]

3. True or False. If false, give a counterexample.
   a) If \(ax \equiv ay \mod m\) and \((m, x) = 1\), then \(x \equiv y \mod m\).

   **Solution:** False. E.g., take \(x = 1\), \(y = 3\), \(a = 2\), \(m = 4\).

   b) If \(ax \equiv ay \mod m\) and \((a, y) = 1\), then \(x \equiv y \mod m\).

   **Solution:** False. E.g., take \(x = 1\), \(y = 3\), \(a = 2\), \(m = 4\).

   c) If \(ax \equiv ay \mod m\), then \(x \equiv y \mod m/(m, x)\).

   **Solution:** False. E.g., take \(x = 1\), \(y = 3\), \(a = 2\), \(m = 4\).

4. Let \(a = 2^83^64^25^36^4\) and \(b = 2^73^24^55^56^5\). Give prime decompositions for the following:
   a) \(ab\)
   b) \([a, b]\)
   c) \((a, b)\)

   **Solution:** We first give prime decompositions for \(a\) and \(b\). Expanding the powers of 4 and 6, we get \(a = 2^{16}3^{10}5^3\) and \(b = 2^{22}3^75^5\). Thus,
   a) \(ab = 2^{16+22}3^{10+7}5^{3+5} = 2^{38}3^{17}5^8\).
   b) \([a, b] = 2^{\max(16,22)}3^{\max(10,7)}5^{\max(3,5)} = 2^{22}3^{10}5^5\).
   c) \((a, b) = 2^{\min(16,22)}3^{\min(10,7)}5^{\min(3,5)} = 2^{16}3^{3}5^3\).
5. Find all solutions or say why none exist:
   a) \(48x + 36y = 20\)

   **Solution:** Here, \((48, 36) = 12\), which does not divide 20, so there are no solutions.

   b) \(48x + 20y = 36\)

   **Solution:** Here, \((48, 20) = 4\) divides 36, so there are solutions. We have
   \[
   48 = 2 \cdot 20 + 8
   \]
   \[
   20 = 2 \cdot 8 + 4.
   \]
   Back substituting, we obtain
   \[
   4 = -2 \cdot 48 + 5 \cdot 20 \quad \text{so}
   \]
   \[
   36 = 9 \cdot 4 = (-18) \cdot 48 + 45 \cdot 20,
   \]
   so \(x_0 = -18\), \(y_0 = 45\) gives one solution. So the general solution is given by
   \[
   x = -18 + n \cdot \frac{20}{4} = -18 + n \cdot 5
   \]
   \[
   y = 45 - n \cdot \frac{48}{4} = 45 - n \cdot 12
   \]
   for all \(n \in \mathbb{Z}\).

6. Suppose \((a, b) = 12\), \([a, b] = 3600\). What are the possible values for \(a\) and \(b\) (as a pair)?

   **Solution:** Here, both \(a\) and \(b\) divide \([a, b] = 2^43^25^2\), so \(a = 2^{r_1}3^{s_1}5^{r_3}\) and \(b = 2^{r_2}3^{s_2}5^{s_3}\), with \(r_i, s_i \in \mathbb{Z}\) for all \(i\).

   We have \((a, b) = 2^23^15^0\), so the formulas for the prime decompositions of \([a, b]\) and \((a, b)\) give
   \[
   \max(r_1, s_1) = 4 \quad \min(r_1, s_1) = 2
   \]
   \[
   \max(r_2, s_2) = 2 \quad \min(r_2, s_2) = 1
   \]
   \[
   \max(r_3, s_3) = 2 \quad \min(r_3, s_3) = 0.
   \]

   We can choose the values of \(r_1\) and \(s_1\), so we take \(r_1 = 4, s_1 = 2\).

   Letting the other values vary we obtain four cases:

   Case 1. \(r_2 = 2, r_3 = 2\). Thus, \(a = 3600\). This forces \(s_2 = 1, s_3 = 0\), so \(b = 12\).

   Case 2. \(r_2 = 2, r_3 = 0\). Thus, \(a = 144\). This forces \(s_2 = 1, s_3 = 2\), so \(b = 300\).

   Case 3. \(r_2 = 1, r_3 = 2\). Thus, \(a = 1200\). This forces \(s_2 = 2, s_3 = 0\), so \(b = 36\).

   Case 4. \(r_2 = 1, r_3 = 0\). Thus, \(a = 48\). This forces \(s_2 = 2, s_3 = 2\), so \(b = 900\).
Exam 1 Solutions

7. Find all non-negative solutions less than the modulus or show why no solutions exist:
   a) $45x \equiv 60 \mod 100$

   Solution: $(100, 45) = 5$ divides 60, so solutions exist. Dividing through by $(100, 45)$, we get
   
   $9x \equiv 12 \mod 20$.

   We first find a multiplicative inverse for $9 \mod 20$. Solving Bezout’s Identity, we get $1 = -4 \cdot 20 + 9 \cdot 9$, so $9 \cdot 9 \equiv 1 \mod 20$. Thus,

   
   $x \equiv 9 \cdot 9x \equiv 9 \cdot 12 \equiv 108 \equiv 8 \mod 20$.
   
   Adding multiples of 20, we see

   
   $x \equiv 8, 28, 48, 68, 88 \mod 100$.

   b) $60x \equiv 45 \mod 100$

   Solution: $(60, 100) = 20$ does not divide 45, so there are no solutions.