1. For $1 you can buy a letter A, B or C. For $2 you can buy a digit 0,\ldots,9. Let \(a_n\) be the number of ways you can spend $\(n\), recording your purchases in order.

a) Give a recurrence relation and initial conditions for calculating \(a_n\).

**Solution:** If the first purchase is a letter, there are \(a_{n-1}\) ways to spend the rest of the money. With three letters available, this gives \(3a_{n-1}\) ways to spend the money if you start with a letter.

If the first purchase is a digit, there are \(a_{n-2}\) ways to spend the rest of the money. With ten digits available, this gives \(10a_{n-2}\) ways to spend the money if you start with a digit. Thus,

\[a_n = 3a_{n-1} + 10a_{n-2}.\]

There is one way to spend $0, so \(a_0 = 1\). There are 3 ways to spend $1, so \(a_1 = 3\). The recurrence relation for \(a_2\) works if we set \(a_0 = 1\), so these do give the correct initial values.

b) Find a formula for \(a_n\).

**Solution:** The associated quadratic is \(x^2 - 3x - 10\), which has roots \(\frac{3\pm\sqrt{9+40}}{2} = -2, 5\). Thus,

\[a_n = a \cdot 5^n + b \cdot (-2)^n\]

for some \(a, b\). Now, \(1 = a_0 = a + b\), so \(b = 1 - a\). Also,

\[3 = a_1 = a \cdot 5 + b \cdot (-2) = a \cdot 5 + (1 - a) \cdot (-2) = a \cdot 7 + (-2).\]

Solving this gives \(a = \frac{5}{7}\), hence \(b = \frac{2}{7}\). Thus,

\[a_n = \frac{5}{7} \cdot 5^n + \frac{2}{7} \cdot (-2)^n\]

Checking this gives \(a_0 = \frac{7}{7} = 1\) and \(a_1 = \frac{21}{7} = 3\).

c) What is \(a_3\)?

**Solution:** \(a_3 = \frac{5}{7} \cdot 5^3 + \frac{2}{7} \cdot (-2)^3 = \frac{5}{7} \cdot 125 + \frac{2}{7} \cdot (-8) = \frac{609}{7} = 87.\)

You could also solve this from

\[a_2 = 3a_1 + 10a_0 = 3 \cdot 3 + 10 = 19\]

\[a_3 = 3a_2 + 10a_1 = 3 \cdot 19 + 10 \cdot 3 = 87.\]
2. Let $a_n$ be the number of ways to roll a die $n$ times with no two consecutive 6’s.

a) Give a recurrence relation and initial conditions for calculating $a_n$.

**Solution:** If the first roll is not a six, there are $a_{n-1}$ ways to finish, giving $5a_{n-1}$ ways in total where the first roll is not six. If the first roll is a six, then the second roll is not six. There are then $a_{n-2}$ ways to finish after the “not six”. Thus, there are $5a_{n-2}$ ways in total where the first roll is a six.

Putting these together, we have

$$a_n = 5a_{n-1} + 5a_{n-2}.$$ 

As usual, there is one way to roll 0 times, so $a_0 = 1$, and $a_1 = 6$, as no outcomes are ruled out if you only roll once. The recursion relation then gives $a_2 = 5 \cdot 6 + 5 \cdot 1 = 35$, which matches the observation that only one of the 36 ways to roll a die twice is not permitted under our rule.

b) Find a formula for $a_n$.

**Solution:** The associated quadratic is $x^2 - 5x - 5$, which has roots $\frac{5 \pm \sqrt{25+20}}{2} = \frac{5 \pm 3\sqrt{5}}{2}$. Thus,

$$a_n = a \cdot \left(\frac{5 + 3\sqrt{5}}{2}\right)^n + b \cdot \left(\frac{5 - 3\sqrt{5}}{2}\right)^n$$

for some $a, b$. Now, $1 = a_0 = a + b$, so $b = 1 - a$. Also,

$$6 = a_1 = a \cdot \frac{5 + 3\sqrt{5}}{2} + (1 - a) \cdot \frac{5 - 3\sqrt{5}}{2}$$

$$= a \left[\frac{5 + 3\sqrt{5}}{2} - \frac{5 - 3\sqrt{5}}{2}\right] + \frac{5 - 3\sqrt{5}}{2}$$

$$= a \cdot 3\sqrt{5} + \frac{5 - 3\sqrt{5}}{2}.$$ 

Solving this gives $a = \frac{1}{3\sqrt{5}} \cdot \frac{7 + 3\sqrt{5}}{2} = \frac{15 + 7\sqrt{5}}{30}$, and $b = 1 - a = \frac{15 - 7\sqrt{5}}{30}$. Thus,

$$a_n = \frac{15 + 7\sqrt{5}}{30} \cdot \left(\frac{5 + 3\sqrt{5}}{2}\right)^n + \frac{15 - 7\sqrt{5}}{30} \cdot \left(\frac{5 - 3\sqrt{5}}{2}\right)^n.$$
3. Let $a_n$ be the number of ways to choose $n$ digits in order so that no two consecutive digits are equal.
   
   a) Give a recurrence relation and initial conditions for calculating $a_n$.
   
   **Solution:** The first $n - 1$ digits can be chosen in $a_{n-1}$ ways. The $n$-th digit can be anything different from the $(n - 1)$-st. So $a_n = 9a_{n-1}$. This derivation assumes that $n - 1$ is at least 1.
   
   The initial condition is that $a_1 = 10$.

   b) Find a formula for $a_n$.
   
   **Solution:** $a_2 = 9 \cdot 10$, $a_3 = 9^2 \cdot 10$, ..., $a_n = 9^{n-1} \cdot 10$.

   c) What is $a_3$?
   
   **Solution:** $a_3 = 9^2 \cdot 10 = 810$.

4. Let $a_n$ be the number of subsets of $\{1, \ldots, n\}$ containing no consecutive integers.
   
   a) Give a recurrence relation and initial conditions for calculating $a_n$.
   
   **Solution:** The number of subsets that do not contain $n$ is $a_{n-1}$.
   
   The subsets that do contain $n$ have the form $S \cup \{n\}$, where $S \subset \{1, \ldots, n-2\}$, since $n-1$ can’t be in any permissible subset that contains $n$. Thus, the number of subsets containing $n$ is $a_{n-2}$. Thus,

   $$a_n = a_{n-1} + a_{n-2}.$$ 

   The initial conditions are that $a_0 = 1$, as the null set only has one subset, while $a_1 = 2$, as both $\{1\}$ and $\emptyset$ are permissible subsets of $\{1\}$.

   b) Find a formula for $a_n$.
   
   **Solution:** The polynomial here is $x^2 - x - 1$, with roots $\frac{1 \pm \sqrt{5}}{2}$. Thus,

   $$a_n = a \cdot \left(\frac{1 + \sqrt{5}}{2}\right)^n + b \cdot \left(\frac{1 - \sqrt{5}}{2}\right)^n.$$

   where $a$ and $b$ are constants determined by the initial conditions.
Exam 2 Solutions

for some $a, b$. Now, $1 = a_0 = a + b$, so $b = 1 - a$. Also,
\[
2 = a_1 = a \cdot \frac{1 + \sqrt{5}}{2} + (1 - a) \cdot \frac{1 - \sqrt{5}}{2}
\]
\[
= a \left[ \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} \right] + \frac{1 - \sqrt{5}}{2}
\]
\[
= a \cdot \sqrt{5} + \frac{1 - \sqrt{5}}{2}.
\]
Solving, we get $a = \frac{1}{\sqrt{5}} \cdot \frac{3 + \sqrt{5}}{2} = \frac{5 + 3\sqrt{5}}{10}$, hence $b = \frac{5 - 3\sqrt{5}}{10}$, so
\[
a_n = \frac{5 + 3\sqrt{5}}{10} \cdot \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{5 - 3\sqrt{5}}{10} \cdot \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]