1. Solve: \( a_n = a_{n-1} + 12a_{n-2} \), with \( a_0 = 0 \), \( a_1 = 1 \).

**Solution:** The characteristic equation is
\[
x^2 - x - 12 = (x - 4)(x + 3),
\]
so the general solution is
\[
a_n = c_1 \cdot 4^n + c_2 (-3)^n
\]
for some \( c_1, c_2 \). The initial conditions give
\[
0 = a_0 = c_1 + c_2, \quad \text{so } c_2 = -c_1.
\]
\[
1 = a_1 = 4c_1 - 3c_2 = 7c_1,
\]
so \( c_1 = \frac{1}{7} \) and \( c_2 = -\frac{1}{7} \). Thus,
\[
a_n = \frac{1}{7} \cdot 4^n - \frac{1}{7} (-3)^n.
\]

2. Urn A contains 8 red balls and 2 blue balls. Urn B contains 3 red balls and 7 blue balls. Flip two coins. If you get two heads, draw from Urn A. Otherwise, draw from Urn B.
   a) What is the probability the ball is red?
   b) If the ball is red, what is the probability it came from Urn A?

**Solution:** We have the following tree diagram:
Thus, the solutions to a) and b) are

\[ P(\text{Red}) = \frac{1}{4} \cdot \frac{8}{10} + \frac{3}{4} \cdot \frac{3}{10} = \frac{17}{40} \]

\[ P(A|\text{Red}) = \frac{P(A \cap \text{Red})}{P(\text{Red})} = \frac{\frac{1}{4} \cdot \frac{8}{10}}{\frac{17}{40}} = \frac{8}{17} \]

3. A group of 100 people were asked to taste 5 flavors of ice cream:
   - Each flavor appealed to 60 people.
   - For each pair of flavors, 30 people liked both.
   - For each three flavors, 10 people liked all of them.
   - For each four flavors, 5 people liked all of them.
   - One person liked all five flavors.

How many people liked none of the flavors?

**Solution**: Let \( A_i \) be the set of people who like the \( i \)-th flavor. Then

\[ |A_1 \cup \cdots \cup A_5| \]

\[ = \sum_{i=1}^{5} |A_i| - \sum_{1 \leq i < j \leq 5} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq 5} |A_i \cap A_j \cap A_k| 
- \sum_{1 \leq i < j < k < l \leq 5} |A_i \cap A_j \cap A_k \cap A_l| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| 
= 60 \cdot \binom{5}{1} - 30 \cdot \binom{5}{2} + 10 \cdot \binom{5}{3} - 5 \cdot \binom{5}{4} + 1 
= 60 \cdot 5 - 30 \cdot 10 + 10 \cdot 10 - 5 \cdot 5 + 1 = 76. \]

Thus,

\[ |(A_1 \cup \cdots \cup A_5)^c| = 100 - 76 = 24. \]

4. Consider all strings of 6 distinct digits (the first may be a 0).
   a) How many such strings are there? **Solution**: \( P(10, 6) \).
   b) How many strings contain a 5 and a 7?
      **Solution**: First pick slots for the 5 and 7. Then fill the other slots:
      \[ P(6, 2) \cdot P(8, 4) . \]
   c) How many strings contain a 5 and a 7 next to each other?
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Solution: First order the 5 and 7, then pick a position for the pair, then fill the other slots:

\[ 2 \cdot 5 \cdot P(8, 4). \]

d) How many contain a 5 and a 7 separated by two other digits?
Solution: First make a block of four: position the 5 and 7 in the outer slots of the block, then fill the inner slots. Then pick a position for the block. Then fill the other two slots:

\[ 2 \cdot P(8, 2) \cdot 3 \cdot P(6, 2). \]

5. How many ways can you seat six people around a table so that Frank and George don’t sit next to each other?

Solution: First seat Frank and look at the others in relation to that seat. There are now three positions possible for George. Pick one. Now seat the other four:

\[ 3 \cdot 4! = 72. \]

6. A committee contains 6 women and 4 men. How many ways can you choose a 5-person subcommittee that has more women than men?

Solution: \( \binom{6}{3} \binom{4}{2} + \binom{6}{4} \binom{4}{1} + \binom{6}{5} \).

7. How many ways can you arrange 6 women and 4 men in a line so that no two men stand next to each other?

Solution: First order the women. (There are 6! ways to do that.) That makes 7 slots into which you could put a man:

\_W\_W\_W\_W\_W\_W\

There are \( P(7, 4) \) ways to arrange the men in those slots. Thus, the solution is

\[ 6! \cdot P(7, 4). \]