Math 220 **Final Exam** **Fall '08**

1. Let $A = \begin{bmatrix} 1 & -2 & 2 & -1 & 2 & -1 & 2 \\ -2 & 5 & -7 & 0 & -3 & 2 & -3 \\ 2 & -5 & 7 & 1 & 4 & -4 & 4 \\ 0 & -2 & 6 & 5 & -1 & -2 & -1 \\ 1 & -5 & 11 & 7 & 1 & -5 & 1 \end{bmatrix}$.

Free gift: The augmented matrix $[A|b]$ reduces to

$$\left[\begin{array}{ccccccc|l} 1 & 0 & -4 & 0 & 9 & -11 & 9 & 5b_1 + 7b_2 + 5b_3 \\ 0 & 1 & -3 & 0 & 3 & -4 & 3 & 2b_1 + 3b_2 + 2b_3 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 & b_2 + b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4b_1 + b_2 - b_3 + b_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5b_1 + b_2 - 2b_3 + b_5 \end{array}\right].$$

a) Write Col(A) as the nullspace of another matrix.
b) What are the rank and nullity of A?
c) Find a basis for Row(A).
d) Find a basis for N(A).
e) Find a basis for Col(A) consisting of columns of A.
f) Write each of the columns not in the basis you gave for Col(A) as a linear combination of the basis elements.
g) What is the general solution of $Ax = y$ for $y = \begin{bmatrix} -1 \\ 8 \\ -10 \\ -14 \\ -23 \end{bmatrix}$?
h) With y as in part g), write y as a linear combination of the basis elements of from part e). (Hint: set the nonpivot variables equal to 0.)
i) Display an invertible matrix P with the property that $P[A|b]$ is the displayed reduction.

2. Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & -2 \\ -1 & 3 & 1 \end{bmatrix}$.

a) What is the first row of A^{adj}?
b) Use it to calculate $\det A$.

3. Let $A = \begin{bmatrix} 23 & 70 & 10 \\ -9 & -26 & -2 \\ 8 & 23 & 2 \end{bmatrix}$. Then $\mathrm{ch}_A(\lambda) = (\lambda+2)^2(\lambda-3)$.
   a) For each eigenvalue of A, find a basis for the associated eigenspace.
   b) Is A diagonalizable? If not, say why not. If so, find a matrix P such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

4. Let $T : R^2 \to R^2$ be the linear transformation
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 3x_2 \\ 2x_1 - x_2 \end{bmatrix}.$$
   a) What is the matrix for T with respect to the standard basis $\mathcal{E} = \{e_1, e_2\}$ of R^2?
   b) Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the basis given by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. What is the matrix $[T]_{\mathcal{B}}$ of T with respect to $\mathcal{B}$?

5. Let $W = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix} \right\}$.
   a) Find an orthonormal basis for W.
   b) Find an orthonormal basis for $W^{\perp}$.
   c) Let T be the orthogonal projection of R^3 onto $W^{\perp}$. What is the matrix of T?

6. Let $T : P_2 \to P_2$ be given by $T(p) = p(5 - 2x)$.
   a) What is the matrix of T with respect to the standard basis $\mathcal{E} = \{1, x, x^2\}$?
   b) What is $\det(T)$?
   c) What is the trace of T?
   d) What is the characteristic polynomial $\mathrm{ch}_T(\lambda)$?
   e) Give a basis for each eigenspace of T.
   f) Is T diagonalizable? If so, find a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal, and display the diagonal matrix $[T]_{\mathcal{B}}$.