1. Let \( A = \begin{bmatrix} 68 & 3 & -51 & 39 \\ -72 & -4 & 54 & -42 \\ 147 & 6 & -112 & 87 \\ 81 & 3 & -63 & 50 \end{bmatrix} \).

a) What is the characteristic polynomial \( \chi_A(\lambda) \)?

**Solution:** We use the Maple command

\[
\text{factor(charpoly}(A, \lambda));
\]

obtaining

\[
\chi_A(\lambda) = (\lambda + 1)^2(\lambda - 2)^2.
\]

b) For each eigenvalue of \( A \), find a basis for the associated eigenspace.

**Solution:** The eigenvalues are \(-1\) and \(2\).

The eigenspace of \(-1\) is the nullspace of \( A + I \). Applying the Maple command “nullspace” to the matrix \( A + I \) (which may be input by copying and pasting \( A \), and manually adding \( I \) to it), we see that \( N(A + I) \) has basis \( v_1, v_2 \), where

\[
v_1 = \begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix}
\]

and

\[
v_2 = \begin{bmatrix} -1 \\ 10 \\ 0 \\ 1 \end{bmatrix}.
\]

The eigenspace of \(2\) is the nullspace of \( A - 2I \). Using the same methods, Maple shows that \( N(A - 2I) \) has basis \( v_3, v_4 \), where

\[
v_3 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 1 \\ \frac{4}{3} \end{bmatrix}
\]

and

\[
v_4 = \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ -\frac{5}{3} \end{bmatrix}.
\]

c) Is \( A \) diagonalizable? If so, find a matrix \( P \) such that \( P^{-1}AP \) is diagonal, and display the diagonal matrix \( P^{-1}AP \).

**Solution:** We set \( P = [v_1 \mid v_2 \mid v_3 \mid v_4] = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -6 & 10 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{4}{3} & -\frac{5}{3} \end{bmatrix} \).

Since the columns of \( P \) form a basis of \( \mathbb{R}^4 \) consisting of eigenvectors of \( A \), \( P^{-1}AP \) is a diagonal matrix whose diagonal entries are the eigenvalues associated (in order) with the columns...
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of $P$: $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. This can be verified via

the Maple command “multiply(inverse($P$),$A$,$P$);”.

2. Let $A = \begin{bmatrix} -17 & 0 & 10 & -5 \\ 22 & 3 & -13 & 9 \\ -45 & 0 & 28 & -15 \\ -30 & 0 & 20 & -12 \end{bmatrix}$.

a) What is the characteristic polynomial $\text{ch}_A(\lambda)$?

**Solution:** Using the same methods as in problem 1, we obtain

$\text{ch}_A(\lambda) = (\lambda + 2)^2(\lambda - 3)^2$.

Thus, the eigenvalues are $-2$ and $3$.

b) For each eigenvalue of $A$, find a basis for the associated eigenspace.

**Solution:** The eigenspace for $-2$ is the nullspace of $A + 2I$. Using Maple as in problem 1, we see $\text{N}(A+2I)$ has basis $v_1$, $v_2$,

where $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$.

The eigenspace for 3 is the nullspace for $A - 3I$. This time, we find that the nullspace is one-dimensional, with basis $v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

c) Is $A$ diagonalizable? If so, find a matrix $P$ such that $P^{-1}AP$ is diagonal, and display the diagonal matrix $P^{-1}AP$.

**Solution:** The algebraic multiplicity of the eigenvalue 3 is two, but its geometric multiplicity (the dimension of its eigenspace) is one. Since these are different, $A$ can’t be diagonalizable.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation

$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 - 7x_2 + 4x_3 \\ 5x_1 + 4x_2 - 2x_3 \\ x_1 - 2x_2 + 4x_3 \end{bmatrix}$.
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a) What is the matrix for $T$ with respect to the standard basis $\mathcal{E} = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$?

SOLUTION: We can read off the matrix of $T$ with respect to the standard basis from the coefficients of the equation for $T$:

$$[T]_{\mathcal{E}} = \begin{bmatrix} 3 & -7 & 4 \\ 5 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

b) Let $\mathcal{B} = \{v_1, v_2, v_3\}$ be the basis given by $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix}$. What is the matrix $[T]_{\mathcal{B}}$ of $T$ with respect to $\mathcal{B}$?

SOLUTION:

$$[T]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{E}}[T]_{\mathcal{E}}[I]_{\mathcal{E}\mathcal{B}} = P^{-1}[T]_{\mathcal{E}}P,$$

where

$$P = [v_1 \mid v_2 \mid v_3] = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & -1 & 6 \end{bmatrix}$$

Since we computed $[T]_{\mathcal{E}}$ in part a), we can find $P^{-1}[T]_{\mathcal{E}}P$ using Maple. We get

$$[T]_{\mathcal{B}} = \begin{bmatrix} 4 & -77 & 109 \\ 1 & 50 & -64 \\ 0 & 33 & -43 \end{bmatrix}.$$
4. Let \( T : P_2 \rightarrow P_2 \) be given by \( T(p) = p(7 - 5x) \).

a) What is the matrix of \( T \) with respect to the standard basis \( \mathcal{E} = \{1, x, x^2\} \)?

**Solution:** To find \([T]_{\mathcal{E}}\), we’ll need to find \( T(1) \), \( T(x) \) and \( T(x^2) \). For a polynomial \( p \), \( T(p) \) is what we get if we substitute \( 7 - 5x \) in place of \( x \) in the formula for \( p \). Thus, \( T(1) = 1 \) (\( x \) doesn’t appear at all in the formula for the constant polynomial \( 1 \)). We also have \( T(x) = 7 - 5x \), and \( T(x^2) = (7 - 5x)^2 = 49 - 70x + 25x^2 \). By definition,

\[
[T]_{\mathcal{E}} = \begin{bmatrix}
1 & 7 & 49 \\
0 & -5 & -70 \\
0 & 0 & 25
\end{bmatrix}.
\]

b) What is \( \det(T) \)?

**Solution:** \( \det(T) = \det([T]_{\mathcal{E}}) \). Since \([T]_{\mathcal{E}}\) is upper triangular, its determinant is the product of its diagonal entries: \( \det(T) = -125 \).

c) What is the trace of \( T \)?

**Solution:** \( \text{tr}(T) = \text{tr}([T]_{\mathcal{E}}) \). The trace of any matrix is the sum of its diagonal entries, so \( \text{tr}(T) = 21 \).

d) What is the characteristic polynomial \( \text{ch}_T(\lambda) \)?

**Solution:**

\[
\text{ch}_T(\lambda) = \text{ch}_{[T]_{\mathcal{E}}}(\lambda) = \det(\lambda I - [T]_{\mathcal{E}}) = \begin{vmatrix}
\lambda - 1 & -7 & -49 \\
0 & \lambda + 5 & 70 \\
0 & 0 & \lambda - 25
\end{vmatrix}.
\]

Since this matrix is upper triangular, its determinant is the product of its diagonal entries, so

\[
\text{ch}_T(\lambda) = (\lambda - 1)(\lambda + 5)(\lambda - 25).
\]

Thus, the eigenvalues of \( T \) are \( 1 \), \( -5 \) and \( 25 \).

e) Give a basis for each eigenspace of \( T \).

**Solution:** We use Maple to find bases for the eigenspaces of \( A = [T]_{\mathcal{E}} \), and then find polynomials whose coordinates with respect to \( \mathcal{E} \) are these basis vectors.

For the eigenvalue \( 1 \), the eigenspace of \( A \) has basis \[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.
\]

This corresponds to the constant polynomial \( 1 \), so the eigenspace of \( T \) with respect to \( 1 \) has basis \( p_1 = 1 \).
For the eigenvalue $-5$, the eigenspace of $A$ has basis $\begin{bmatrix} -\frac{7}{6} \\ 1 \\ 0 \end{bmatrix}$, which corresponds to the polynomial $-\frac{7}{6} + x$. Thus, the eigenspace for $T$ with respect to $-5$ has basis $p_2 = -\frac{7}{6} + x$. (We could also use $7 - 6x$, a multiple of this $p_2$.)

For the eigenvalue $25$, the eigenspace of $A$ has basis $\begin{bmatrix} \frac{49}{36} \\ -\frac{7}{3} \\ 1 \end{bmatrix}$. This corresponds to the polynomial $p_3 = \frac{49}{36} - \frac{7}{3}x + x^2$. (As above, we could replace this with $49 - 84x + 36x^2$.)

Note: You can check directly that the polynomials $p_1$, $p_2$ and $p_3$ are eigenvectors for the stated eigenvalues. For $p_1$, this is given by our calculation of $T(1)$. In the other cases, you can use Maple’s “expand” command for polynomial arithmetic. E.g., for $p_2$, use

\[
\text{expand}\left(-\frac{7}{6} + (7 - 5 \times x) + 5 \times (-\frac{7}{6} + x)\right); 
\]

This computes $T(p_2) + 5p_2$, and the output of 0 verifies that $p_2$ is an eigenvector for $T$ with eigenvalue $-5$.

f) Is $T$ diagonalizable? If so, find a basis $B$ such that $[T]_B$ is diagonal, and display the diagonal matrix $[T]_B$.

**Solution:** $T$ is diagonalizable because $\text{ch}_T(\lambda)$ factors completely over $\mathbb{R}$ and each eigenvalue has algebraic multiplicity 1. Any basis $B$ consisting of eigenvectors will have the property that $[T]_B$ is diagonal, with the diagonal entries being the eigenvalues associated to the basis elements (in the order they appear in the basis). Thus, take $B = p_1, p_2, p_3$, which gives

\[
[T]_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 25 \end{bmatrix}.
\]