Math 220  Exam 2 Solutions  Fall ’08

1. Let $A = \begin{bmatrix} 1 & -1 & -1 & -1 & 0 & -6 & 3 \\ -1 & 2 & 5 & 0 & -1 & 14 & 3 \\ 0 & 1 & 4 & -1 & 0 & 3 & 2 \\ 1 & -1 & -1 & -1 & 1 & -11 & -1 \\ 1 & -2 & -5 & 0 & 1 & -14 & -3 \end{bmatrix}$.

Free gift: The augmented matrix $[A|b]$ reduces to

$$
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -3 & 5 & b_1 + 2b_2 + 3b_3 - 2b_4 + 4b_5 \\
0 & 1 & 4 & -1 & 0 & 3 & 2 & 2b_3 - b_4 + b_5 \\
0 & 0 & 0 & 0 & 1 & -5 & -4 & -b_1 - 2b_2 + b_3 - b_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_2 - b_3 + b_4 - 2b_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & b_2 + b_5
\end{bmatrix}.
$$

a) Write $\text{Col}(A)$ as the nullspace of another matrix.

Solution: $b \in \text{Col}(A)$ if and only if the expressions in the last two rows of the augmentation column of the reduction are zero. Thus,

$$
\text{Col}(A) = N \left( \begin{bmatrix} 0 & -1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \right)
$$

b) What are the rank and nullity of $A$?

Solution: The rank is the number of pivots: 3. The nullity is the dimension of the nullspace: 4.

c) Find a basis for $\text{Row}(A)$.

Solution: We take the nonzero rows of the reduction of $A$:

$$
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -3 & 5 \\
0 & 1 & 4 & -1 & 0 & 3 & 2 \\
0 & 0 & 0 & 1 & -5 & -4
\end{bmatrix}
$$

d) Find a basis for $\text{N}(A)$.

Solution: We set the free variables equal to arbitrary real numbers: $x_3 = s$, $x_4 = t$, $x_6 = u$, $x_7 = v$ and then set $b = 0$. 


and solve. We get
\[
x = s \begin{bmatrix}
-3 \\
-4 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ t \begin{bmatrix}
2 \\
1 \\
0 \\
1 \\
0 \\
0 \\
0
\end{bmatrix}
+ u \begin{bmatrix}
3 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
+ v \begin{bmatrix}
-3 \\
-2 \\
1 \\
0 \\
1 \\
0 \\
0
\end{bmatrix}
= sw_1 + tw_2 + uw_3 + vw_4.
\]

Here, \(w_1, \ldots, w_4\) are defined by the equation above.

e) Find a basis for \(\text{Col}(A)\) consisting of columns of \(A\).

**SOLUTION:** Write \(v_i\) for the \(i\)th column of \(A\). Then \(\text{Col}(A)\) has
a basis given by the pivot columns of \(A\): \(v_1, v_2, v_5\). Thus, the
basis is given by
\[
\begin{bmatrix}
1 \\
-1 \\
0 \\
1 \\
1
\end{bmatrix},
\begin{bmatrix}
-1 \\
2 \\
1 \\
1 \\
-2
\end{bmatrix},
\begin{bmatrix}
0 \\
-1 \\
1 \\
1 \\
1
\end{bmatrix}.
\]

f) Write each of the columns not in the basis you gave for \(\text{Col}(A)\)
as a linear combination of the basis elements.

**SOLUTION:**
\[
Aw_1 = 0 \quad \text{says} \quad v_3 = 3v_1 + 4v_2.
\]
\[
Aw_2 = 0 \quad \text{says} \quad v_4 = -2v_1 - v_2.
\]
\[
Aw_3 = 0 \quad \text{says} \quad v_6 = -3v_1 + 3v_2 - 5v_5.
\]
\[
Aw_4 = 0 \quad \text{says} \quad v_7 = 5v_1 + 2v_2 - 4v_5.
\]

g) What is the general solution of \(Ax = y\) with
\[
y = \begin{bmatrix}
2 \\
-5 \\
-1 \\
4 \\
5
\end{bmatrix}?
\]

**SOLUTION:** We set the coordinates of \(y\) equal to \(b_1, \ldots, b_5\) and
then reduce, evaluating the expressions in the augmentation
column of the reduction. Thus, we are solving
\[
\begin{bmatrix}
1 & 0 & 3 & -2 & 0 & -3 & 5 & 1 \\
0 & 1 & 4 & -1 & 0 & 3 & 2 & -1 \\
0 & 0 & 0 & 1 & -5 & -4 & 2 & |
\end{bmatrix}
\]
As in part d) we set \(x_3 = s, x_4 = t, x_6 = u, x_7 = v\) and obtain
\[
x = \begin{bmatrix}
1 \\
-1 \\
0 \\
0 \\
2 \\
0 \\
0
\end{bmatrix} + sw_1 + tw_2 + uw_3 + vw_4,
\]
with \(w_1, \ldots, w_4\) as in d).

h) With \(y\) from part g), write \(y\) as a linear combination of the basis elements from part e).
SOLUTION: Setting \(s = t = u = v = 0\) in the solution to f), we obtain a solution to \(Ax = y\), Calculating \(Ax\), we see
\(y = v_1 - v_2 + 2v_5\).
i) Display an invertible matrix \(P\) with the property that \(P[A|b]\) is the displayed reduction.
SOLUTION: The reduction given above is \(P[A|b] = [PA|Pb]\), so the last column of the reduction is \(Pb\) this lets us read off the matrix \(P\):
\[
P = \begin{bmatrix}
1 & 2 & 3 & -2 & 4 \\
0 & 0 & 2 & -1 & 1 \\
-1 & -2 & 1 & 0 & -1 \\
0 & -1 & -1 & 1 & -2 \\
0 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

2. Let \(p_1 = 3 + 2x - x^2, p_2 = 2 + x + 3x^2,\) and \(p_3 = -2 - x - 2x^2\).
a) Show that \(p_1, p_2, p_3\) form a basis for \(P_2\).
SOLUTION: Since \(P_2\) has dimension 3, it suffices to show \(p_1, p_2, p_3\) span \(P_2\). Thus, we wish to set
\[
c_1p_1 + c_2p_2 + c_3p_3 = a_0 + a_1x + a_2x^2
\]
with \(a_0, a_1, a_2\) arbitrary and solve for \(c_1, c_2, c_3\). Expanding the linear combination \(c_1p_1 + c_2p_2 + c_3p_3\) as a polynomial and
equating coefficients, we see that
\[
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix}
\]
is a solution of
\[
\begin{bmatrix}
  3 & 2 & -2 \\
  2 & 1 & -1 \\
  -1 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2
\end{bmatrix}.
\]
Solving, we get
\[
\begin{align*}
  c_1 &= -a_0 + 2a_1 \\
  c_2 &= -5a_0 + 8a_1 + a_2 \\
  c_3 &= -7a_0 + 11a_1 + a_2.
\end{align*}
\]
b) Find the coordinates of \( \mathbf{p} = -1 + x + 2x^2 \) relative to \( \mathcal{B} = \{ \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \} \).

**Solution:** Using the solution to part a), we get
\[
\mathbf{p} = 3\mathbf{p}_1 + 15\mathbf{p}_2 + 20\mathbf{p}_3,
\]
so
\[
[\mathbf{p}]_\mathcal{B} = \begin{bmatrix} 3 \\ 15 \\ 20 \end{bmatrix}.
\]