

A bag contains 5 red balls and 8 green ones. You draw twice **without** replacement.

1. What is the probability the first is red or the second is green?

SOLUTION:

$$\begin{aligned} P(\text{1st R or 2nd G}) &= P(\text{1st R}) + P(\text{2nd G}) - P(\text{1st R and 2nd G}) \\ &= P(\text{1st R}) + P(\text{2nd G}) - P(\text{2nd G} | \text{1st R})P(\text{1st R}) \\ &= \frac{5}{13} + \frac{8}{13} - \frac{8}{12} \cdot \frac{5}{13}. \end{aligned}$$

2. What is the probability the two draws are of different colors?

SOLUTION: If they are of different colors, then either the first is red and the second green or the first is green and the second red. Note that these two cases are mutually exclusive. Thus,

$$P(\text{two draws different}) = P(\text{1st R and 2nd G}) + P(\text{1st G and 2nd R}).$$

We saw in Problem 1 that

$$P(\text{1st R and 2nd G}) = \frac{8}{12} \cdot \frac{5}{13}.$$

Similarly,

$$P(\text{1st G and 2nd R}) = P(\text{2nd R} | \text{1st G})P(\text{1st G}) = \frac{5}{12} \cdot \frac{8}{13}.$$

Thus,

$$P(\text{two draws different}) = \frac{8}{12} \cdot \frac{5}{13} + \frac{5}{12} \cdot \frac{8}{13}.$$

Deal 5 cards from a well-shuffled deck.

3. What is the probability they are all spades?

SOLUTION:

$$\begin{aligned} P(\text{all } \spadesuit) &= P(\text{5th } \spadesuit | \text{1st 4 } \spadesuit)P(\text{4th } \spadesuit | \text{1st 3 } \spadesuit)P(\text{3rd } \spadesuit | \text{1st 2 } \spadesuit) \\ &\quad \cdot P(\text{2nd } \spadesuit | \text{1st } \spadesuit)P(\text{1st } \spadesuit) \\ &= \frac{9}{48} \cdot \frac{10}{49} \cdot \frac{11}{50} \cdot \frac{12}{51} \cdot \frac{13}{52}. \end{aligned}$$

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4. What is the probability they are all in the same suit?

SOLUTION:

$$\begin{aligned} P(\text{all same suit}) &= P(\text{all } \spadesuit) + P(\text{all } \heartsuit) + P(\text{all } \diamondsuit) + P(\text{all } \clubsuit) \\ &= 4 \cdot P(\text{all } \spadesuit) \\ &= 4 \cdot \frac{9}{48} \cdot \frac{10}{49} \cdot \frac{11}{50} \cdot \frac{12}{51} \cdot \frac{13}{52}. \end{aligned}$$

5. Roll a pair of dice. What is the probability you got doubles given that the sum is 10?

SOLUTION: There are three ways to get a sum of 10: (4, 6), (5, 5), (6, 4). Each is equally likely, and only one of them is doubles. So the conditional probability is $\frac{1}{3}$.

6. Roll a pair of dice. What is the probability you got doubles or a sum of 5?

SOLUTION: "Doubles" and "sum of 5" are mutually exclusive, since doubles always have an even sum. So

$$\begin{aligned} P(\text{doubles or sum of 5}) &= P(\text{doubles}) + P(\text{sum of 5}) \\ &= \frac{6}{36} + \frac{4}{36}. \end{aligned}$$

7. Deal two cards from a well shuffled deck. What is the probability the first is a king or the second is a face card (K Q or J)?

SOLUTION: There are 12 face cards, including the kings, so

$$\begin{aligned} P(\text{1st K or 2nd face}) &= P(\text{1st K}) + P(\text{2nd face}) - P(\text{1st K and 2nd face}) \\ &= P(\text{1st K}) + P(\text{2nd face}) \\ &\quad - P(\text{2nd face} \mid \text{1st K})P(\text{1st K}) \\ &= \frac{4}{52} + \frac{12}{52} - \frac{11}{51} \cdot \frac{4}{52}. \end{aligned}$$

8. Deal two cards from a well shuffled deck. What is the probability at least one of them is an ace?

SOLUTION:

$$\begin{aligned} P(\text{1st A or 2nd A}) &= P(\text{1st A}) + P(\text{2nd A}) - P(\text{1st A and 2nd A}) \\ &= P(\text{1st A}) + P(\text{2nd A}) - P(\text{2nd A} \mid \text{1st A})P(\text{1st A}) \\ &= \frac{4}{52} + \frac{4}{52} - \frac{3}{51} \cdot \frac{4}{52}. \end{aligned}$$

9. Roll a pair of dice 5 times. What is the probability the sum is 6 on less than half the tosses?

SOLUTION: To get six on less than half the tosses, you either get 2 sixes, 1 six, or no sixes. The three are mutually exclusive, so

$$\begin{aligned} P(\leq \text{half sixes}) &= P(2 \text{ sixes}) + P(1 \text{ six}) + P(\text{no sixes}) \\ &= \binom{5}{2} \left(\frac{5}{36}\right)^2 \left(\frac{31}{36}\right)^3 + \binom{5}{1} \left(\frac{5}{36}\right)^1 \left(\frac{31}{36}\right)^4 \\ &\quad + \binom{5}{0} \left(\frac{5}{36}\right)^0 \left(\frac{31}{36}\right)^5 \\ &= 10 \cdot \left(\frac{5}{36}\right)^2 \left(\frac{31}{36}\right)^3 + 5 \cdot \left(\frac{5}{36}\right)^1 \left(\frac{31}{36}\right)^4 + \left(\frac{31}{36}\right)^5. \end{aligned}$$

10. An urn contains 57 red balls and three white balls. Draw 15 times with replacement. What is the probability you get at least two white balls?

SOLUTION:

$$\begin{aligned} P(\geq 2 \text{ W}) &= 1 - P(< 2 \text{ W}) \\ &= 1 - [P(1 \text{ W}) + P(0 \text{ W})] \\ &= 1 - \left[\binom{15}{1} \left(\frac{3}{60}\right)^1 \left(\frac{57}{60}\right)^{14} + \binom{15}{0} \left(\frac{3}{60}\right)^0 \left(\frac{57}{60}\right)^{15} \right] \\ &= 1 - 15 \cdot \left(\frac{3}{60}\right)^1 \left(\frac{57}{60}\right)^{14} - \left(\frac{57}{60}\right)^{15} \end{aligned}$$

11. You set up a lotto game with your friends. An entry consists of a choice of 5 numbers from 1 to 30. You win if you match the 5 numbers in that range drawn at random by the game's administrator. (The order of the numbers doesn't matter.)

What is the probability of winning?

SOLUTION: Since any two entries have an equal probability of winning, the probability of winning is the reciprocal of the number of possible entries:

$$\frac{1}{\binom{30}{5}} = \frac{1}{\left(\frac{30!}{5!25!}\right)} = \frac{5!25!}{30!}$$

Exam 2 Solutions

12. In a multiple choice test, you have a choice of four answers for each question. If you guess right, you get 5 points. If you guess wrong, you lose a point. Suppose the test has 20 questions, and a passing score is 35. What is the probability you can pass by guessing?

SOLUTION: The payoff for guessing an answer is like the payoff from drawing at random from

$$\boxed{\begin{array}{|c|c|c|c|} \hline 5 & -1 & -1 & -1 \\ \hline \end{array}}.$$

The average of the box is $\frac{5+3(-1)}{4} = \frac{2}{4} = \frac{1}{2}$, so

$$EV = \frac{1}{2} \cdot 20 = 10.$$

Our formula gives $\sigma(\text{box}) = (5 - (-1))\sqrt{\frac{1}{4} \cdot \frac{3}{4}}$, so

$$SE = 6\sqrt{.25 * .75} \cdot \sqrt{20} \approx 11.62 .$$

Thus,

$$SU(35) \approx \frac{35 - 10}{11.62} \approx 2.15 .$$

From the normal table, the area to the right of 2.15 is 1.58%.

Game: Draw twice with replacement from the following box:

$$\boxed{\begin{array}{|c|c|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline \end{array}}$$

If the sum is 3, you win \$37. Otherwise you lose \$1.

Suppose you play the game 450 times. The next three questions concern your net gain/loss.

SOLUTION: Out of the 81 possible outcomes from drawing twice with replacement from the box, two are winners: (1, 2) and (2, 1), so the payoff from a single play is like drawing once from the box

$$\boxed{\begin{array}{|c|c|c|c|} \hline 2 & 37 & 79 & -1 \\ \hline \end{array}} .$$

The average of this box is $\frac{2(37)+79(-1)}{81} = -\frac{5}{81}$, so

$$EV = -\frac{5}{81} \cdot 450 \approx -27.78 .$$

Our formula gives $\sigma(\text{box}) = (37 - (-1))\sqrt{\frac{2}{81} \cdot \frac{79}{81}}$, so

$$SE = 38\sqrt{\frac{2}{81} \cdot \frac{79}{81}} \cdot \sqrt{450} \approx 125.09 .$$

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13. What is the probability you lose at least \$20?

SOLUTION:

$$SU(-20) = \frac{-20 - EV}{SE} \approx .06 .$$

By the normal table, the area below this point is 52.39%.

14. What is the probability you lose at least \$100?

SOLUTION:

$$SU(-100) = \frac{-100 - EV}{SE} \approx -.58 .$$

By the normal table, the area below this point is 28.10%.

15. What is the probability you win at least \$100?

SOLUTION:

$$SU(100) = \frac{100 - EV}{SE} \approx 1.02 .$$

By the normal table, the area above this point is 15.39%.