A bag contains 5 red balls and 7 green ones. You draw three times without replacement.

1. What is the probability all three are red?
   **Solution:**
   
   \[
P(\text{all 3 R}) = P(\text{3rd R} | \text{first 2 R}) \cdot P(\text{2nd R} | \text{1st R}) \cdot P(\text{1st R})
   \]
   
   \[
   = \frac{3}{10} \cdot \frac{4}{11} \cdot \frac{5}{12}
   \]

2. What is the probability all three are the same color?
   **Solution:** The probability they are all the same color is the sum of the probability all three are red and the probability all three are green.

   \[
P(\text{all 3 G}) = P(\text{3rd G} | \text{first 2 G}) \cdot P(\text{2nd G} | \text{1st G}) \cdot P(\text{1st G})
   \]
   
   \[
   = \frac{5}{10} \cdot \frac{6}{11} \cdot \frac{7}{12}
   \]
   
   Thus,
   \[
P(\text{all same color}) = \frac{3 \cdot 4 \cdot 5 + 5 \cdot 6 \cdot 7}{10 \cdot 11 \cdot 12}.
   \]

3. Roll a pair of dice. What is the probability you got doubles given that the sum is 4?
   **Solution:** There are three ways to get a sum of 4: 1,3; 2,2; 3,1. Only one of the three is doubles. So the conditional probability is \(\frac{1}{3}\).

4. Roll a pair of dice. What is the probability you got doubles or a sum of 8?
   **Solution:** There are five ways to get a sum of 8: 2,6; 3,5; 4,4; 5,3; 6,2. Only one of these is doubles. Thus,

   \[
P(\text{“doubles” or “sum of 8”}) = P(\text{doubles}) + P(\text{sum of 8})
   \]
   
   \[
   - P(\text{“doubles” and “sum of 8”})
   \]
   
   \[
   = \frac{6}{36} + \frac{5}{36} - \frac{1}{36} = \frac{10}{36}.
   \]
5. Deal two cards from a well shuffled deck. What is the probability the first is an ace or the second is a king?
   SOLUTION: We first note that
   \[
P(1\text{st A and 2nd K}) = P(2\text{nd K} | 1\text{st A}) \cdot P(1\text{st A})
   \]
   \[
   = \frac{4}{51} \cdot \frac{4}{52}.
   \]
   Thus,
   \[
P(1\text{st A or 2nd K}) = P(1\text{st A}) + P(2\text{nd K}) - P(1\text{st A and 2nd K})
   \]
   \[
   = \frac{4}{52} + \frac{4}{52} - \frac{4 \cdot 4}{51 \cdot 52}.
   \]

6. Deal two cards from a well shuffled deck. What is the probability at least one of them is the ace of spades?
   SOLUTION:
   \[
P(\text{one is A♠}) = P(1\text{st A♠}) + P(2\text{nd A♠}) - P(\text{both are A♠})
   \]
   \[
   = \frac{1}{52} + \frac{1}{52} - 0.
   \]

You have a biased coin that lands head 65% of the time and lands tails 35% of the time. Flip it 6 times.
7. What is the probability you get exactly 5 heads?
   SOLUTION:
   \[
P(5H) = \binom{6}{5} \cdot (.65)^5 \cdot (.35)^1
   \]
   \[
   = 6 \cdot (.65)^5 \cdot (.35)^1
   \]

8. What is the probability you get more heads than tails?
   SOLUTION:
   \[
P(\text{more H than T}) = P(4H) + P(5H) + P(6H)
   \]
   \[
   = \binom{6}{4} (.65)^4 (.35)^2 + \binom{6}{5} (.65)^5 (.35)^1 + (.65)^6
   \]
   \[
   = 15 \cdot (.65)^4 (.35)^2 + 6 \cdot (.65)^5 (.35)^1 + (.65)^6
   \]
Roll a pair of dice 20 times.

9. What is the probability you get double sixes at least once?

**Solution:** Let E be the event “double sixes”.

\[ P(E \text{ at least once}) = 1 - P(E \text{ never occurs}) \]
\[ = 1 - \left( \frac{35}{36} \right)^{20}. \]

10. What is the probability you get double sixes at least twice?

**Solution:** with E as above,

\[ P(E \text{ at least twice}) = 1 - [P(E \text{ never occurs}) + P(E \text{ occurs once})] \]
\[ = 1 - \left[ \left( \frac{35}{36} \right)^{20} + \binom{20}{1} \left( \frac{1}{36} \right) \left( \frac{35}{36} \right)^{19} \right] \]
\[ = 1 - \left( \frac{35}{36} \right)^{20} - 20 \cdot \left( \frac{1}{36} \right) \left( \frac{35}{36} \right)^{19}. \]

11. You set up a lotto game with your friends. An entry consists of a choice of 6 numbers from 1 to 20. You win if you match the 6 numbers in that range drawn at random by the game’s administrator. (The order of the numbers doesn’t matter.)

What is the probability of winning?

**Solution:** The number of possible entries is

\[ \binom{20}{6} = \frac{20!}{6! \cdot 14!} = 38760. \]

The probability of winning is 1/38760.

12. In a multiple choice test, you have a choice of four answers for each question. If you guess right, you get 4 points. If you guess wrong, you lose a point. Suppose the test has 25 questions, and a passing score is 30. What is the probability you can pass by guessing?

**Solution:** We can model this on drawing 25 times with replacement from

\[ \begin{bmatrix} 4 & 3 & -1 \end{bmatrix} \]

and adding the numbers drawn. The expected value is 25 times the average of the box, or 25/4 = 6.25. The standard error is

\[ SE = \sqrt{25 \cdot (4 - (-1)) \cdot \frac{1}{4} \cdot \frac{3}{4}} \approx 10.825 \]

Thus, the z-value of 30 is (30 − 6.25)/10.825 ≈ 2.19. This has a percentile value of 98.57%, so the probability of passing is 1.43%.
**Exam 2 Solutions**

**Game:** Draw twice with replacement from the following box:

```
1 2 3 4 5 6 7
```

If the sum is 2, you win $45. Otherwise you lose $1.

Suppose you play the game 980 times. The next three questions concern your net gain/loss.

**Solution:** There is 1 chance in 49 of winning, so we can model the gain/loss on drawing 980 times with replacement from

```
45 48 -1
```

The expected value is 980 times the average of the box, or −60. The standard error is

\[ SE = \sqrt{980 \cdot (45 - (-1)) \cdot \sqrt{\frac{1}{49} \cdot \frac{48}{49}}} \approx 203.608 \]

13. What is the probability you lose at least $30?

**Solution:** The \( z \)-value of −30 is \((-30 - (-60))/203.608 \approx .15\). This has a percentile value of 55.96%, which gives the answer.

14. What is the probability you lose at least $100?

**Solution:** The \( z \)-value of −100 is \((-100 - (-60))/203.608 \approx -20\). This has a percentile value of 42.07%, which gives the answer.

15. What is the probability you win at least $100?

**Solution:** The \( z \)-value of 100 is \((100 - (-60))/203.608 \approx .79\). This has a percentile value of 78.52%, so the probability of winning more than $100 is 21.48%.