Another Use of χ^2

The χ^2 distribution may be used to describe the way the S-value of the sample approximates the population SD (σ). Specifically, the relationship between the two is given by

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

where n is the sample size. One then uses the χ^2 table with $n-1$ degrees of freedom to interpret the result.

Example 1. A customer service department wants to change from having separate lines at each window to having a single line for all of them. Their motivation is to increase the consistency in waiting times.

Under the old system, the SD (σ) for the waiting times was 15 minutes. On the first day of the new system, they track 21 randomly selected patrons. For this sample, the S-value is 10 minutes. Is this difference significant?

Solution: Here, $S = 10$ and $\sigma = 15$, so

$$\chi^2 = 20 \cdot (10/15)^2$$

$$\approx 8.89$$

Here, there are 20 degrees of freedom. Looking at the χ^2 table, we see that χ^2 lies in the lower tail, with a P-value less than 5%. Thus, the test is statistically significant, and the new system has been shown to be more consistent than the old one.

Example 2. You make a study of a quantitative variable. With a sample size of 12, the sample S-value is 26.11. Give a 90% confidence interval for the population SD.

Warning: Confidence intervals for SD's are not symmetric around the sample S-value. There is more involved here than computing a radius.

Solution: There are 11 degrees of freedom. The confidence interval is obtained by restricting the values of χ^2 to the interior 90% of the distribution, i.e., by cutting off the 5% tails at each end. Thus, the 90% confidence interval for 11 degrees of freedom is given by setting

$$4.58 < \chi^2 < 19.68$$

We have $s = 26.11$, and we want to estimate σ. Thus, we wish to solve

$$4.58 < \frac{11 \cdot (26.11)^2}{\sigma^2} < 19.68$$

$$4.58 < \frac{7500}{\sigma^2} < 19.68$$

Thus,

$$\frac{1}{19.68} < \frac{\sigma^2}{7500} < \frac{1}{4.58}$$

$$\frac{7500}{19.68} < \sigma^2 < \frac{7500}{4.58}$$

$$381.1 < \sigma^2 < 1637.6$$

$$19.52 < \sigma < 40.47$$

Thus, the confidence interval is (19.52, 40.47). Note that this is indeed not symmetric around the sample S-value of 26.11.

Exercises:

1. Theory predicts that the SD for a variable should be 10. You take a random sample of size 15. The sample S-value is 7. What is the P value for this experiment? Do the results contradict the theory?

2. Theory predicts that the SD for a variable should be 20. You take a random sample of size 9. The sample S-value is 25. What is the P value for this experiment? Do the results contradict the theory?

3. You make a study of a quantitative variable. With a sample size of 8, your sample has an average of 125 and an S-value of 10. Give 98% confidence intervals for both the population average and the population SD.