SOLUTIONS/HINTS FOR PRACTICE PROBLEMS IN CHAPTERS 14 AND 16

Note: Most problems in the textbook have brief solutions/hints at the end of the book, which you can use as a guide. For more solutions/hints (to problems not covered in class, during the two revision sessions), see below. Also make sure to revisit all problems solved in class during the two revision sessions.

14.3 First prove that D, E, F are collinear (see Fig. 14.5). Apply the Menelaus theorem to: $\triangle ABV$ with transversal $\overleftrightarrow{A'B'}$, $\triangle BCV$ with transversal $\overleftrightarrow{B'C'}$, and $\triangle CAV$ with transversal $\overleftrightarrow{C'A'}$. Multiply the three relations, and cancel out various numerators and denominators. By rearranging what is left, you obtain exactly the relation in the Menelaus theorem for $\triangle ABC$ proving the collinearity of D, E, F.

For the reciprocal, assume that D, E, F are collinear. Let V be the intersection of $\overleftrightarrow{AA'}$ and $\overleftrightarrow{BB'}$. The fact that $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ are concurrent amounts to showing that C, C', V are collinear. But this is precisely what we proved above, when applied to $\triangle EAA'$ and $\triangle DBB'$.

14.4 Apply the Menelaus theorem three times, for $\triangle PQR$ with transversals $\overleftrightarrow{AF}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{DE}$ (see Fig. 14.8). Multiply the three relations, and observe that you obtain the three fractions in the Menelaus theorem for $\triangle PQR$ proving the collinearity of L, M, N.

However, you get several extra factors which need to be cancelled out. To apply the hint, write the Menelaus theorem for $\triangle PQR$ with transversals $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$.

14.7 Straighforward from Ceva's theorem.

14.9 Solve only if you know the law of sines, which you need to apply to $\triangle AEF$, $\triangle BDF$, and $\triangle CED$, where D, E, F are the three (supposedly) collinear points.

14.15 See the hint in the book, which should suffice.

14.18 See Fig. 14.13 and the hint in the book. For this purpose, write a system of equations in the unknown segments, using several times the fact that the two segments determined by a random point P outside a circle and the two points where the tangents from P meet the circle have equal lengths.

14.17 Use the information from 14.18 before.

14.19 Use the exterior bisector theorem, stated as Theorem 14.9.

14.20 Note that the considered lines through the midpoints are the bisectors of the medial triangle (as the sides of the latter are parallel to the sides of the original triangle). This shows that they are concurrent (at the incenter of the medial triangle).

Question: how can the intersection (incenter of the medial triangle) be expressed in terms of the incenter of the original triangle via a dilation?

14.23 Not very relevant for this exam, as it uses mostly information about quadrilaterals inscribed in circles.

14.24 See Fig. 14.12. By definition $\overleftrightarrow{AI}$ is the bisector of the interior angle at A, and it passes through I_a as well. Also, $\overleftrightarrow{I_bI_c}$ is the bisector of the exterior angle at A. It follows that $\overleftrightarrow{AI}$ is perpendicular to $\overleftrightarrow{I_bI_c}$ (why?). The rest is immediate, including the statement about the nine point circle (why?).

14.25 Note first that it suffices to show that the following lines are concurrent: the bisector from A and the lines joining the midpoints N, P of $\overline{AB}$, $\overline{AC}$ with the opposite tangency points F, E of the excircles of the medial triangle. For this purpose, apply the Ceva theorem to $\triangle ABC$. Let Q be the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{PE}$, and R the intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{NF}$. Use the similarity of $\triangle QNE$ and $\triangle QAP$, as well as of $\triangle RFP$ and $\triangle RNA$. Also use the information from problem 14.18 above related to the segments determined by the excircles.

16.3 More or less done in lectures.

16.4 Very similar to the corresponding statement in the plane. Prove that the composite $\sigma_\Omega\sigma_\Delta\sigma_\Gamma$ is a reflection (under the given assumptions), by rewriting $\sigma_\Delta\sigma_\Gamma$ such that you make σ_Ω cancel out in the above composite. Then use the fact that $(\sigma_\Omega\sigma_\Delta\sigma_\Gamma)^2$ is the identity, in order to derive the given relation (multiply both sides by certain reflections).

16.6 Check the answers in the book and try to justify them. For instance, for (e): define a screw first; if true, start with a screw, decompose it as a product of reflections, and regroup them to obtain a product of two rotations; if false, give a solid justification of why this can never happen.

16.22 Various rotations about lines joining two centers of faces, two midpoints of opposite parallel sides, and diagonals. Make sure you get a total of 24 isometries, including the identity. In fact, the group of even isometries of the cube is isomorphic to the group of permutations of $\{1, 2, 3, 4\}$, which has $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$ elements; so not just the cardinalities match up, but also the structure! (This can be seen by identifying an isometry with the corresponding permutation of the 4 diagonals of the cube.)

16.24 $x' = 2a - x$, $y' = 2b - y$, $z' = 2c - z$. See material covered in class related to inversions in n-space.