SKEW SCHUBERT POLYNOMIALS

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Abstract. We define skew Schubert polynomials to be normal form (polynomial) representatives of certain classes in the cohomology of a flag manifold. We show that this definition extends a recent construction of Schubert polynomials due to Bergeron and Sottile in terms of certain increasing labeled chains in Bruhat order of the symmetric group. These skew Schubert polynomials expand in the basis of Schubert polynomials with nonnegative integer coefficients that are precisely the structure constants of the cohomology of the complex flag variety with respect to its basis of Schubert classes. We rederive the construction of Bergeron and Sottile in a purely combinatorial way, relating it to the construction of Schubert polynomials in terms of rc-graphs.

Introduction

Skew Schur polynomials $S_{\lambda/\mu}(x_1,\ldots,x_k)$ play an important and multi-faceted role in algebraic combinatorics. For example, they are generating functions for the Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu}$

$$S_{\lambda/\mu}(x_1,\ldots,x_k) \;=\; \sum_{\nu} c^{\lambda}_{\mu,\nu}\, S_\nu(x_1,\ldots,x_k)\,. \tag{1}$$

They are also generating functions for Young tableaux of shape λ/μ

$$S_{\lambda/\mu}(x_1,\ldots,x_k) \;=\; \sum \mathbf{x}^T\,, \tag{2}$$

where the sum is over all Young tableaux T of shape λ/μ and $\mathbf{x}^T$ is a monomial associated to T. The relationship between these two very different facets of skew Schur polynomials involves the combinatorics of Young tableaux [20, Chapter 7, Appendix 1]. Young tableaux can be thought of as increasing labeled chains in Young's lattice, with covers labeled by pairs (k,l), where k is the entry in the box corresponding to the cover, and l the content of that box (that is, the difference between its column and row); labels are ordered lexicographically. We introduce skew Schubert polynomials, which are the Schubert polynomial analogs of skew Schur polynomials, in the sense that they generalize the two properties mentioned above.

In Section 1, we introduce our key concepts of increasing labeled chains in the Bruhat order, which generalize Young tableaux, and of skew Schubert polynomials, $\mathfrak{S}_{w/u}(x)$ for permutations $u\leq w$. In particular, we show that skew Schubert polynomials are generating functions both for Littlewood-Richardson coefficients $c^w_{u,v}$ for Schubert polynomials (the analog of (1)), and

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for increasing chains in the Bruhat order (the analog of (2)). Our skew Schubert polynomials are different from those defined in [10] (using skew divided difference operators), which do not expand with nonnegative coefficients in the basis of Schubert polynomials, in general.

The skew Schubert polynomial $\mathfrak{S}_{w_0/w}(x)$ is the ordinary Schubert polynomial $\mathfrak{S}_w(x)$, and our formula for $\mathfrak{S}_{w_0/w}(x)$ in terms of increasing chains is the formula of Corollary 5.3 in [4], which inspired us (see the next section for the notation). In Section 2, we give a purely combinatorial proof of that formula, relating it to a standard construction [7, 5, 8] in terms of rc-graphs by giving a content-preserving bijection between rc-graphs and increasing chains.

1. Skew Schubert Polynomials

Let Σ_n be the symmetric group of permutations of $\{1, 2, \ldots, n\}$. For $w \in \Sigma_n$, the *length*, $\ell(w)$, of w is the number of inversions of w. Set $w_0 \in \Sigma_n$ to be the longest permutation, $w_0 := n \ldots 2\,1$. We use basics on Schubert and Schur polynomials, which may be found in any of [14, 15, 9, 16].

The main outstanding problem in the theory of Schubert polynomials is the Littlewood-Richardson problem [21, Problem 11]: Determine the structure constants $c^w_{u,v}$ defined by the polynomial identity

$$\mathfrak{S}_u(x) \cdot \mathfrak{S}_v(x) \ = \ \sum_w c^w_{u,v} \mathfrak{S}_w(x)\,.$$

Since every Schur polynomial is a Schubert polynomial, this problems asks for the analog of the classical Littlewood-Richardson rule. The classical Littlewood-Richardson coefficients $c^\lambda_{\mu,\nu}$ have important and intricate combinatorial properties [20, Chapter 7, Appendix 1]. The Littlewood-Richardson coefficients $c^w_{u,v}$ for Schubert polynomials should be similarly important. Indeed, they are intersection numbers of Schubert varieties; more precisely, $c^w_{u,v}$ enumerates flags in a suitable triple intersection of Schubert varieties indexed by u, v, and w_0w.

The cohomology classes σ_w (called Schubert classes) for $w \in \Sigma_n$ of Schubert varieties form an integral basis for the cohomology ring $H^*\mathbb{F}\ell_n$ of the flag manifold. Many identities and some formulas for the constants $c^w_{u,v}$ have been obtained by studying the class $\sigma_u \cdot \sigma_{w_0w}$ in $H^*\mathbb{F}\ell_n$ [18, 2, 3]. This is because, for $u, v, w \in \Sigma_n$, we have the following identity in the cohomology ring

$$\sigma_u \cdot \sigma_{w_0w} \cdot \sigma_v \ = \ c^w_{u,v}\,\sigma_{w_0}$$

and hence, by the duality of the intersection pairing

$$\sigma_u \cdot \sigma_{w_0w} \ = \ \sum_{v\in\Sigma_n} c^w_{u,v}\,\sigma_{w_0v}\,. \tag{3}$$

With this motivation, we would like to define the skew Schubert polynomial $\mathfrak{S}_{w/u}(x)$ for permutations $u \leq w$ in Σ_n to be *the* polynomial representative of the class $\sigma_u \cdot \sigma_{w_0w}$. Unfortunately, the cohomology of the flag manifold is isomorphic to the quotient ring

$$H^*\mathbb{F}\ell_n \ := \ \mathbb{Z}[x_1, x_2, \ldots, x_n]/\langle e_i(x_1, \ldots, x_n) \mid i = 1, \ldots, n\rangle\,,$$

and so there is no well-defined polynomial representative of $\sigma_u \cdot \sigma_{w_0w}$. Here $e_i(x_1, \ldots, x_n)$ is the ith elementary symmetric polynomial in $x_1, \ldots, x_n$.

If however we choose the degree reverse lexicographic term order on monomials in the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$, with $x_1 < x_2 < \cdots < x_n$, then every element of this quotient ring has a unique normal form polynomial representative [22, Prop. 1.1] with respect to this term order. These normal form representatives are elements of

$$\mathbb{Z}\cdot\{x^\alpha \mid \alpha=(\alpha_1,\ldots,\alpha_n) \text{ with } \alpha_i\leq n-i\}\,.$$

Equivalently, if we set $\delta := (n-1,\ldots,2,1,0)$, then normal form representatives of cohomology classes are sums of monomials dividing x^δ.

Fomin, Gelfand, and Postnikov [6] observed that for $w\in\Sigma_n$, the Schubert polynomial $\mathfrak{S}_w(x)$ is the normal form representative of the corresponding Schubert class σ_w. In fact, this feature of Schubert polynomials, that one does not need to work modulo an ideal, is what led Lascoux and Schützenberger to their definition of Schubert polynomial. Thus we define the *skew Schubert polynomial* $\mathfrak{S}_{w/u}(x)$ to be the normal form representative of the class $\sigma_u\cdot\sigma_{w_0w}$. These skew Schubert polynomials are generating functions for the coefficients $c^w_{u,v}$.

Theorem 1. *For $u\leq w$ in Σ_n, we have*

$$\mathfrak{S}_{w/u}(x) \;=\; \sum_v c^w_{u,v}\,\mathfrak{S}_{w_0v}(x)\,. \tag{4}$$

Proof. Each side of (4) is the normal form representative of the corresponding side of (3). □

While $\mathfrak{S}_{w/u}(x)$ depends upon n, the Laurent polynomial $x^{-\delta}\mathfrak{S}_{w/u}(x)$ does not, as $\mathfrak{S}_u(x)$ and $x^{-\delta}\mathfrak{S}_{w_0w}(x)$ are each independent of n. The independence of $\mathfrak{S}_u(x)$ is the familiar stability property of Schubert polynomials. The independence of $x^{-\delta}\mathfrak{S}_{w_0w}(x)$ may be similarly deduced using divided differences. This also follows from Proposition 3, which we prove in Section 2.

The skew Schubert polynomial $\mathfrak{S}_{w/u}(x)$ is also a generating function for certain chains from u to w in the Bruhat order. The Bruhat order is defined by its covers; $u\lessdot w$ if and only if $u^{-1}w$ is a transposition (k,l) with $u(k)<u(l)$, and for every $k<i<l$ we have either $u(i)<u(k)$ or $u(l)<u(i)$. Thus $u\lessdot u\cdot(k,l)$ with $k<l$ if $u(k)<u(l)$ and at no position between k and l does u take a value between $u(k)$ and $u(l)$. This implies that the difference in lengths $\ell(w)-\ell(u)$ equals 1.

The *labeled Bruhat order* is the labeled réseau (labeled directed multigraph) obtained from the Bruhat order by drawing a directed edge $u\xrightarrow{(k,b)}w$ whenever $u\lessdot w$ with $u^{-1}w=(i,j)$, where $i\leq k<j$ and $b=u(i)=w(j)$. We note that all the constructions and statements still hold if we uniformly set $b=u(j)=w(i)$, but it is more convenient to use the first definition. There will be $j-i$ such directed edges for every cover. This structure was defined in [1] and has been crucial in subsequent work on the problem of multiplying Schubert polynomials by Bergeron and Sottile. In [2] this structure was called the colored Bruhat order.

To any (saturated) chain γ in this réseau, we associate a monomial x^γ in the variables $x_1,\ldots,x_{n-1}$, where the power of x_i counts how often i was the first coordinate of a label in γ. A chain

$$u_0 \xrightarrow{(k_1,b_1)} u_1 \xrightarrow{(k_2,b_2)} \cdots \xrightarrow{(k_m,b_m)} u_m \tag{5}$$

in this réseau is *increasing* if its sequence of labels is increasing in the lexicographic order on pairs of integers.

These definitions allow a nice reformulation of a combinatorial construction of Schubert polynomials given in [4].

Theorem 2. *Let $u \leq w$ be permutations in Σ_n. Then*

$$\mathfrak{S}_{w/u}(x) \;=\; \sum x^\delta/x^\gamma, \tag{6}$$

the sum over all increasing chains γ in the labeled Bruhat order from u to w.

Note that $\mathfrak{S}_w(x) = \mathfrak{S}_{w_0/w}(x)$. We recover the formula of Bergeron and Sottile for $\mathfrak{S}_w(x)$ [4]. Denote by $\Gamma(w, w_0)$ the set of all increasing labeled chains from w to w_0.

Proposition 3 (Bergeron-Sottile [4])**.** *Let $w \in \Sigma_n$. Then*

$$\mathfrak{S}_w(x) \;=\; \sum_{\gamma\in\Gamma(w,w_0)} x^\delta/x^\gamma.$$

Theorem 2 has an immediate enumerative consequence. The *type* α of a chain γ in the labeled Bruhat order is the (weak) composition α whose ith component counts the number of occurrences of i as the first coordinate of an edge label. Thus $x^\gamma = x^\alpha$. For $u \leq w$ in the Bruhat order and a composition α of $\ell(w) - \ell(u)$, let $I_\alpha(u, w)$ count the number of increasing chains from u to w of type α. Since $\mathfrak{S}_{w_0v}(x) = \mathfrak{S}_{w_0/w_0v}(x)$, equating coefficients in (4) and using Theorem 2 we obtain

Corollary 4. $\qquad I_\alpha(u, w) \;=\; \sum_v c^w_{u,v} I_\alpha(w_0v, w_0).$

For a composition α with $n-1$ parts where $\alpha \leq \delta$ coordinatewise, set

$$h_\alpha(x) \;:=\; h_{\alpha_1}(x_1)h_{\alpha_2}(x_1, x_2)\cdots h_{\alpha_{n-1}}(x_1, \ldots, x_{n-1})\,,$$

where $h_a(x_1, \ldots, x_k)$ is the complete homogeneous symmetric polynomial of degree a in the variables $x_1, \ldots, x_k$. We let h_a and h_α denote the corresponding cohomology classes in $H^*\mathbb{F}\ell_n$. For $f \in H^*\mathbb{F}\ell_n$, set $\psi_\alpha(f)$ to be the coefficient of σ_{w_0} in the product $f \cdot h_\alpha$. This gives a linear map $\psi_\alpha \colon H^*\mathbb{F}\ell_n \to \mathbb{Z}$.

Lemma 5. *For any $f \in H^*\mathbb{F}\ell_n$, $\psi_\alpha(f)$ is the coefficient of the monomial x^δ/x^α in the normal form representative of f.*

Proof. This follows from two observations. First, both ψ_α and the map associating to f the coefficient of the monomial x^δ/x^α in the normal form representative of f are $\mathbb{Z}$-linear maps on $H^*\mathbb{F}\ell_n$. Second, the lemma holds on the basis of Schubert classes, whose representatives are Schubert polynomials. This was shown by Kirillov and Maeno [11] and is also a consequence of the Pieri formula [18] and the construction of Schubert polynomials given in Proposition 3. $\square$

Proof of Theorem 2. Given a chain γ in the Bruhat order, let $\mathrm{end}(\gamma)$ denote its end point. The Pieri-type formula for Schubert classes [18] is

$$\sigma_u \cdot h_a \;=\; \sum \sigma_{\mathrm{end}(\gamma)}(x)\,, \tag{7}$$

the sum over all increasing chains that begin at u, have length a, and whose labels have first coordinate k.

Applying Lemma 5 to the class $\sigma_u \cdot \sigma_{w_0w}$ shows that the coefficient of the monomial x^δ/x^γ in the normal form representative of $\sigma_{w/u}$ is the coefficient of σ_{w_0} in the triple product $\sigma_u \cdot \sigma_{w_0w} \cdot h_\alpha$, where α is the type of the chain γ. Since the coefficient of σ_{w_0} in a product $\sigma_v \cdot \sigma_{w_0w}$ is the

Kronecker delta δ_w^v (by the duality of the intersection pairing), this is the coefficient of σ_w in the product $\sigma_u \cdot h_\alpha$. The theorem now follows by expanding this product and iteratively applying the Pieri formula. $\square$

Remark 6. By the definition of skew Schubert polynomials, we have

$$\mathfrak{S}_{w_0/w_0w}(x) \;=\; \mathfrak{S}_{w_0w}(x) \;=\; \mathfrak{S}_{w/1}(x)\,,$$

since $w_0w_0 = 1$ and $\mathfrak{S}_1(x) = 1$. Thus there should exist a natural bijection between $\Gamma(1, w)$ and $\Gamma(w_0w, w_0)$ that preserves type. Corollary 4 suggests that, for $u \leq w$, this bijection should generalize to a map from $\Gamma(u, w)$ to the union $\coprod_v \Gamma(w_0v, w_0)$ such that the cardinality of the inverse image of each chain in $\Gamma(w_0v, w_0)$ is $c^w_{u,v}$. This would give a combinatorial interpretation for $c^w_{u,v}$, and thus solve the Littlewood-Richardson problem.

Remark 7. The skew Schubert polynomial $\mathfrak{S}_{w/u}(x)$ also has a geometric interpretation as the representative in cohomology of a skew Schubert variety

$$X_uF_\bullet \,\cap\, X_{w_0w}F'_\bullet\,,$$

where $X_wF_\bullet$ is the Schubert variety whose representative in cohomology is $\mathfrak{S}_w(x)$ and $F_\bullet$ and $F'_\bullet$ are flags in general position. Stanley noted that skew Schur polynomials have a similar geometric-cohomological interpretation [19, §3].

Remark 8. Schubert polynomials $\mathfrak{S}_w(x)$ when w has a single descent at position k (w is a Grassmannian permutation) are Schur symmetric polynomials in $x_1, \ldots, x_k$. Even if w and u are Grassmannian permutations with the same descent, then it is not necessarily the case that the skew Schubert polynomial $\mathfrak{S}_{w/u}(x)$ equals the corresponding skew Schur polynomial. We illustrate this in Σ_4, using partitions and skew partitions for indices of Schur polynomials.

We have the Schubert/Schur polynomials

$$\begin{aligned}
\mathfrak{S}_{1324}(x) &\;=\; x_1 + x_2 &&\;=\; S_{(1)}(x_1, x_2)\,, \qquad \text{and}\\
\mathfrak{S}_{2413}(x) &\;=\; x_1^2x_2 + x_1x_2^2 &&\;=\; S_{(2,1)}(x_1, x_2)\,,
\end{aligned}$$

and also the skew Schur polynomial

$$\begin{aligned}
S_{(2,1)/(1)}(x_1, x_2) &\;=\; x_1x_2 \;+\; x_1x_2 + x_1^2 + x_2^2\\
&\;=\; S_{(1,1)}(x_1, x_2) \;+\; S_{(2)}(x_1, x_2) \;=\; \mathfrak{S}_{2314}(x) + \mathfrak{S}_{1423}(x)\,.
\end{aligned}$$

On the other hand, $\mathfrak{S}_{2413/1324}(x) = \mathfrak{S}_{1324}(x) \cdot \mathfrak{S}_{w_02413}(x)$ and $w_02413 = 3142$. By Monk's formula (or the Pieri formula (7)), we compute $\mathfrak{S}_{2413/1324}(x)$ to be

$$\begin{aligned}
\mathfrak{S}_{1324}(x) \cdot \mathfrak{S}_{3142}(x) &\;=\; \mathfrak{S}_{3241}(x) \;+\; \mathfrak{S}_{4132}(x) \;+\; \mathfrak{S}_{3412}(x)\\
&\;=\; \mathfrak{S}_{w_02314}(x) + \mathfrak{S}_{w_01423}(x) + \mathfrak{S}_{w_02143}(x)\,.
\end{aligned}$$

The first two terms of $\mathfrak{S}_{2413/1324}(x)$ carry the same information as the two terms of $S_{(2,1)/(1)}(x_1, x_2)$, but there is also a third term. This is because the multiplication of Schubert polynomials is richer than the multiplication of Schur polynomials. By Theorem 1, this expansion of $\mathfrak{S}_{2413/1324}(x)$ records the fact that $\mathfrak{S}_{2413}(x)$ is a summand (with coefficient 1) in each of

$$\mathfrak{S}_{1324}(x) \cdot \mathfrak{S}_{2314}(x)\,, \qquad \mathfrak{S}_{1324}(x) \cdot \mathfrak{S}_{1423}(x)\,, \qquad \text{and} \qquad \mathfrak{S}_{1324}(x) \cdot \mathfrak{S}_{2143}(x)\,.$$

Note that the last product (unlike the first two) is not symmetric in x_1 and x_2 even though both $\mathfrak{S}_{1324}(x)$ and $\mathfrak{S}_{2413}(x)$ are symmetric in x_1 and x_2.

2. RC-Graphs and Increasing Labeled Chains

RC-graphs are combinatorial objects associated to permutations w in the symmetric group Σ_n. They were defined in [7, 1] as certain subsets of

$$\{(k,b)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\mid k+b\leq n\}\,.$$

We may linearly order such a subset of pairs by setting

$$(k,b)\leq(j,a)\quad\Longleftrightarrow\quad(k<j)\ \text{ or }\ (k=j\ \text{ and }\ b\geq a)\,.$$

Let (k_i,b_i) be the ith pair in this linear order. Consider the sequence

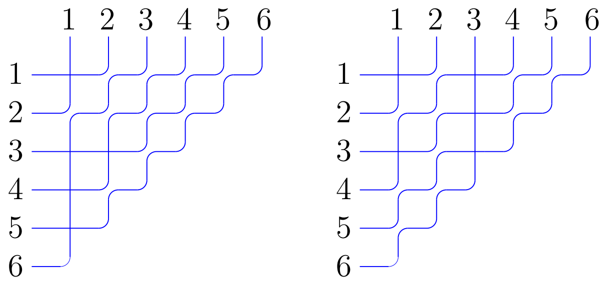
$$d(R)\ :=\ (k_1+b_1-1,k_2+b_2-1,\ldots)\,.$$

An rc-graph R is *associated* to a permutation $w\in\Sigma_n$ if $d(R)$ is a reduced decomposition of w. We denote the collection of rc-graphs associated to a permutation w by $\mathcal{R}(w)$, and the permutation corresponding to a given rc-graph R by $w(R)$.

We represent an rc-graph R as a pseudoline diagram recording the history of the inversions of w. To this end, draw n pseudolines going up and to the right such that the ith pseudoline begins at position $(i,1)$ and ends at position $(1,w(i))$. (The positions are numbered as in a matrix, as illustrated in Example 9 below.) The rule for constructing the pseudoline diagram is the following: two pseudolines entering at position (i,j) cross at that position if (i,j) is in R, and otherwise avoid each other at that position. Note that two pseudolines cross at most once, as $d(R)$ is reduced.

Conversely, a pseudoline diagram represents a permutation w, where $w(i)$ is the endpoint of the pseudoline beginning at position $(i,1)$. Given a pseudoline diagram where no two pseudolines cross more than once, the crossings give an rc-graph associated to the permutation represented by the pseudoline diagram.

Example 9. Here are the pseudoline diagrams of two rc-graphs associated to the permutation $w=215463$.



Given an rc-graph R, define the monomial

$$x^R\ :=\ \prod_{(i,j)\in R}x_i\,.$$

As shown in [5, 7], the Schubert polynomial $\mathfrak{S}_w$ indexed by w can be expressed as

$$\mathfrak{S}_w\ =\ \sum_{R\in\mathcal{R}(w)}x^R\,. \tag{8}$$

Comparing this expression with Proposition 3 suggests that there should be a natural bijection between $\mathcal{R}(w)$ and $\Gamma(w, w_0)$, where $x^R \cdot x^\gamma = x^\delta$, when R corresponds to γ. Indeed, we give such a bijection.

Given an rc-graph R in $\mathcal{R}(w)$, we greedily construct the sequence of rc-graphs $R = R_0, R_1, \ldots, R_{\ell(w_0)-\ell(w)}$ as follows. Given R_i, add the pair (k, b) to R_i where (k, b) is the pair not in R_i that is minimal in the lexicographic order on pairs. We have the following lemma.

Lemma 10. *Given $R \in \mathcal{R}(w)$, construct the sequence $R = R_0, R_1, \ldots, R_{\ell(w_0)-\ell(w)}$ as above. Then*

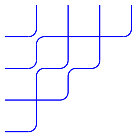
(1) *Every subset R_i is an rc-graph.*
(2) *$w = w(R_0) \lessdot w(R_1) \lessdot \ldots \lessdot w(R_{\ell(w_0)-\ell(w)}) = w_0$ is a saturated chain in Bruhat order.*
(3) *If R_{i+1} is obtained from R_i by adding the pair (k, b), then in the labeled Bruhat order we have the labeled cover*

$$w(R_i) \xrightarrow{(k,b)} w(R_{i+1})\,.$$

(4) *The labeled chain with the labels of* (3) *is increasing.*

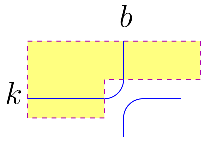
Example 11. Here is an rc-graph, its pseudoline diagram, and the associated increasing labeled chain.

$$\begin{array}{ccc} \cdot & + & + \\ \cdot & \cdot & \\ + & & \end{array}$$



$$1432 \xrightarrow{(1,1)} 4132 \xrightarrow{(2,1)} 4231 \xrightarrow{(2,2)} 4321$$

Proof of Lemma 10. We prove the first statement by induction. Suppose that R_i is an rc-graph and we add the pair (k, b) to obtain R_{i+1}, where (k, b) is the minimal pair not in R_i. Consider the pseudoline diagram of R_i near the position (k, b):



By construction, R_i contains every crossing in the shaded region, so the two pseudolines at this position never cross, with one connecting k to b as drawn. Adding the crossing gives a new pseudoline diagram for a permutation with exactly one more inversion. Indeed, assume that we create a double crossing between one of the two pseudolines obtained from the drawn ones by adding the extra crossing and another pseudoline. It is easy to check graphically that this can only happen if the latter pseudoline meets the pseudoline connecting k to b in R_i twice. But this is impossible, as R_i is an rc-graph.

These arguments show that $w(R_i) \lessdot w(R_{i+1})$, implying the second statement. For the third, note that (k, b) is a possible label, and the fourth follows by our greedy algorithm for adding pairs not in R. □

Observe that the increasing chain $\gamma(R)$ constructed in Lemma 10 from an rc-graph R has the property that

$$\text{if } u \xrightarrow{(k,b)} v \text{ is a labeled cover in } \gamma(R), \text{ then } u^{-1}v = (k,l), \text{ for some } l > k. \tag{9}$$

A priori, given that $u \xrightarrow{(k,b)} v$ is a cover, we are only guaranteed that $u^{-1}v = (j,l)$ with $j \leq k < l$. We prove that this property holds for any increasing chain ending in w_0, and also that the map $\gamma \mapsto \gamma(R)$ is a bijection from $\mathcal{R}(w)$ to $\Gamma(w, w_0)$. This will immediately imply Proposition 3.

Theorem 12. *Let $w \in \Sigma_n$ and γ be an increasing chain from w to w_0. Then*

(1) *γ has property* (9).
(2) *The set of pairs (k,b) with $k + b \leq n$ and (k,b) not a label of a cover in γ is an rc-graph.*

Proof. We prove these statements by downward induction on $\ell(w)$. They hold trivially when $w = w_0$. Consider the first two steps in γ:

$$w \xrightarrow{(j,a)} v \xrightarrow{(k,b)} \cdots \tag{10}$$

Since γ is increasing, we have $j < k$ or $j = k$ and $a < b$. Since the part of the chain beginning with v is an increasing chain, our inductive hypothesis implies that subsequent covers involve only positions of v greater than or equal to k. Thus if $i < k$, we must have $v(i) = n + 1 - i$ and $v(k) = b$.

Suppose that $w^{-1}v = (i,l)$ so that $w(i) = a = v(l)$ and $i \leq j < l$. We consider cases $i = k$, $i = k - 1$, and $i < k - 1$ for i separately, showing that in each case, we have $j = i$.

If $i = k$, then, as $i \leq j \leq k$, we must have $j = k = i$.

If $i < k - 1$, then the betweenness condition for covers in the Bruhat order and the values of v on $1, 2, \ldots, k-1$ force $l = i + 1$ and so again we have $j = i$.

Now suppose that $i = k - 1$ and $j \neq i$ so that $j = k$ and we then have $a < b$. Then

$$v(i) \;=\; v(k-1) \;=\; n+2-k \;>\; v(k) \;=\; b \;>\; a \;=\; v(l)\,,$$

and so the betweenness condition on covers $w < v$ is violated. This completes the proof of Statement 1.

For the second statement, consider the procedure of successively removing labels (k,b) of covers of γ from the rc-graph of w_0 (which contains all crossings). Suppose that this process applied to the chain (10) above v has created an rc-graph R associated to v. We argue that removing the crossing (j,a) from R creates an rc-graph associated to w. First, (j,a) is a crossing of R, as we have only removed crossings that exceed (j,a) in the lexicographic order. Consider the pseudoline diagram of R near (j,a), which we represent on the left.

Note that R contains all crossings in the shaded region. Removing the crossing at (j,a) creates the picture on the right. Arguing as in the proof of Lemma 10 (but in reverse) shows that we obtain an rc-graph associated to w. $\square$

Remark 13. Property (9) of increasing chains from w to w_0 implies that the branching in the tree of increasing chains from w to w_0 is quite simple; it only depends upon the permutation at a node and not on the history of the chain. The branches at a permutation u consists of all possible covers $u \xrightarrow{(k,b)} u(k,l)$, where k is the minimal position i such that $u(i)+i < n+1$ and $b = u(i)$ and l is any position greater than k such that $u(l) > b = u(k)$ and if $k < i < l$, then $u(i)$ is not between $u(k)$ and $u(l)$.

We may order the branches at u by the position l. This leads to an efficient lexicographic search of this tree for generating all such increasing chains, and thus all rc-graphs in $\mathcal{R}(w)$ as well as the multiset of monomials in the Schubert polynomial $\mathfrak{S}_w(x)$. The computational complexity and memory usage of this algorithm depend only upon n (the number of entries in w), the length $l(w)$ of w, and the number of increasing chains, $c := \mathfrak{S}_w(1,1,\ldots,1)$. This is true, in fact, for any algorithm generating c combinatorial objects corresponding to the monomials in the Schubert polynomial $\mathfrak{S}_w(x)$. Furthermore, it is not hard to see that the above algorithm is of order $O(nlc)$, where $l := l(w_0) - l(w) = \binom{n}{2} - l(w)$. Indeed, note that the number of internal nodes of a rooted tree with a fixed number of leaves (c in our case) and a fixed distance from its root to all its leaves (l in our case) is maximized by the tree with all internal nodes having only one descendant; thus, there are $O(lc)$ nodes. On the other hand, finding the descendants of a given node is of order $O(n)$.

Let us also note that a closely related algorithm, with the same complexity as the one above, appears in [12, 17]; it is described in terms of removing crossings from the rc-graph corresponding to w_0. More precisely, the authors of the mentioned papers use combinatorial versions of the divided difference operators in order to recursively generate the rc-graphs corresponding to a permutation w. One starts from w_0 and applies these operators using any chain in the weak order on Σ_n from w_0 to w. A certain choice of chain guarantees that each rc-graph in the obtained tree has at least one descendant upon applying the corresponding operator. It turns out that the rc-graphs on a given level i in this tree correspond (via the bijection described above) to the subchains starting at level $l-i$ in the tree constructed by the previous algorithm.

While this algorithm for generating the multiset of monomials in the Schubert polynomial $\mathfrak{S}_w(x)$ also computes this polynomial, A. Buch has pointed out to us that the transition formula of Lascoux and Schützenberger [14] provides a more efficient algorithm to *compute* $\mathfrak{S}_w(x)$.

We conclude by mentioning the connection with the insertion algorithm for rc-graphs in [13], which bijectively proves the Pieri formula (7) for multiplying a Schubert polynomial by a complete homogeneous symmetric polynomial $h_i(x_1,\ldots,x_k)$. Given an rc-graph R with $x^R = x_1^{\alpha_1}\ldots x_{n-1}^{\alpha_{n-1}}$, the chain $\gamma(R)$ can be obtained by successive insertions into R corresponding to multiplications of x^R by the monomials $x_k^{n-k-\alpha_k}$ in $h_{n-k-\alpha_k}(x_1,\ldots,x_k)$, for $k=1,\ldots,n-1$. In this this special case, the insertion algorithm reduces to the older one in [1].

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