A K-THEORY VERSION OF MONK'S FORMULA AND SOME RELATED MULTIPLICATION FORMULAS

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This paper is dedicated to Camelia.

Abstract. We derive an explicit formula, with no cancellations, for expanding in the basis of Grothendieck polynomials the product of two such polynomials, one of which is indexed by an arbitrary permutation, and the other by a simple transposition; hence, this is a Monk-type formula, expressing the hyperplane section of a Schubert variety in K-theory. Our formula is in terms of increasing chains in the k-Bruhat order on the symmetric group with certain labels on its covers. An intermediate result concerns the multiplication of a Grothendieck polynomial by a single variable. As applications, we rederive some known results, such as Lascoux's transition formula for Grothendieck polynomials. Our results are reformulated in the context of recently introduced Pieri operators on posets and combinatorial Hopf algebras. In this context, we derive an inverse formula to the Monk-type one, which immediately implies a new formula for the restriction of a dominant line bundle to a Schubert variety.

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1. Introduction

Grothendieck polynomials are representatives for *Schubert classes* in the K-theory of the variety of complete flags in $\mathbb{C}^n$. They were introduced by Lascoux and Schützenberger in [15], and studied in more detail in [12]. These polynomials are generalizations of *Schubert polynomials*, which represent Schubert classes in cohomology, and were introduced by Lascoux and Schützenberger in [14]. Both Schubert and Grothendieck polynomials are indexed by permutations in the symmetric group. An important open problem in algebraic combinatorics is to describe combinatorially the expansion of a product of two Schubert polynomials in the basis of Schubert polynomials; the corresponding coefficients are known as *Littlewood-Richardson coefficients*. In the simplest case, one of the polynomials is indexed by a simple transposition. The corresponding multiplication rule is known as *Monk's formula* [20]; in the context of generalized flag varieties, corresponding to arbitrary Lie groups, it is known as *Chevalley's formula* [5]. Monk's formula highlights the importance of a suborder of the *Bruhat order* on the symmetric group, known as the *k-Bruhat order*. Subsequent work on the Littlewood-Richardson coefficients confirmed the strong relationship between them and the combinatorics of the k-Bruhat order (see [3]). For instance, the *Pieri formula* for Schubert polynomials (which is concerned with

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the multiplication of such a polynomial by an elementary or complete homogeneous symmetric polynomial) is expressed in terms of chains in the k-Bruhat order (see the next section).

In this paper we generalize Monk's formula to Grothendieck polynomials, thus obtaining a formula for the hyperplane section of a Schubert variety in K-theory. As expected, the k-Bruhat order plays a central role again, although the generalized formula is more complicated, in the sense that it involves chains of arbitrary length in the k-Bruhat order (rather than just length 1, as in the classical Monk formula). It is interesting to compare these chains with those in the Pieri formula for Schubert polynomials (see below). In fact, there is a common generalization of the Monk-type formula in this paper and the Pieri formula for Schubert polynomials, which is a Pieri-type formula for Grothendieck polynomials [17]. Let us also note that none of the formulas mentioned above involves cancellations; furthermore, all formulas except the last, most general one, are multiplicity free.

An intermediate result in deriving the generalized Monk formula is a formula for multiplying a Grothendieck polynomial by a single variable, which appears in Section 3. A corollary to this formula is Lascoux's *transition formula* for Grothendieck polynomials in [13].

In Section 5 we reformulate our two multiplication rules for Grothendieck polynomials in the context of *Pieri operators* on posets. These operators were recently introduced in [2], and one of the main applications was to a new framework for the Littlewood-Richardson problem for Schubert polynomials. In our context, Pieri operators prove to be useful too, allowing us to easily derive a formula which is, in a certain sense, the "inverse" of the generalized Monk formula. In fact, the inversion is best seen in the more general context of *combinatorial Hopf algebras* defined in [1], where it is just the convolution inverse in a certain Hopf algebra. We conclude by deriving a corollary to our inverse formula, which turns out to be a formula for the restriction of a dominant line bundle to a Schubert variety. This formula is no longer multiplicity free, but, on the other hand, it involves no negative terms.

There are a couple of related results in the literature. A Monk-type formula for Grothendieck polynomials already appeared in [12]. In a sense, it is more general than ours, because it involves double Grothendieck polynomials, and is thus relevant to the T-equivariant K-theory of the flag variety. However, the mentioned formula generally involves cancellations, unlike ours. Let us also note that a different formula for the restriction of a dominant line bundle to a Schubert variety appeared in [9]. It involves Young tableaux and a construction in terms of *keys* of tableaux. This formula was later generalized in [21] to flag varieties corresponding to arbitrary complex semisimple Lie groups, based on *Littelmann paths.* It is not clear how to relate the formulas in terms of tableaux and Littelmann paths to our formula in terms of chains in the k-Bruhat order. Although the existing formulas are in terms of double Grothendieck polynomials, unlike ours, there are two advantages to the latter: (1) it immediately gives the Monk-type formula by the inversion procedure mentioned above, whereas it is not clear how to invert the other two formulas; (2) it is an identity in an algebra of formal power series, not just for Schubert classes in K-theory.

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2. Background

In this section we give a brief introduction to the theory of Schubert and Grothendieck polynomials; for more information, we refer the reader to [18, 19, 8, 12].

Let Fl_n be the variety of complete flags $(0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n)$ in $\mathbb{C}^n$. This is an irreducible algebraic variety of complex dimension $\binom{n}{2}$. Its integral cohomology ring $H^*(Fl_n)$ is isomorphic to $\mathbb{Z}[x_1, \ldots, x_n]/I_n$, where I_n is the ideal generated by symmetric functions in $x_1, \ldots, x_n$ with constant term 0; here, the elements x_i are identified with the Chern classes of the dual line bundles L_i^*, where $L_i := V_i/V_{i-1}$ are tautological line bundles. The variety Fl_n is a disjoint union of cells indexed by permutations w in the symmetric group S_n. Their closures are the *Schubert varieties* X_w, of complex dimension $l(w)$; here $l(w)$ denotes the length of w, that is, the number of its inversions. The Schubert polynomial $\mathfrak{S}_w(x)$ (defined below) is a certain polynomial representative for the cohomology class corresponding to X_w. It is a homogeneous polynomial in $x_1, \ldots, x_{n-1}$ of degree $l(w)$ with nonnegative integer coefficients.

The K-theory $K^0(Fl_n)$ of the flag variety is the Grothendieck ring of complex vector bundles over Fl_n under direct sum and tensor product. We have that $K^0(Fl_n)$ is isomorphic to the same ring as $H^*(Fl_n)$; but this time we identify x_i with the K-theory Chern class $1 - 1/y_i$ of the line bundle L_i^*, where $y_i := 1/(1 - x_i)$ represents L_i in the Grothendieck ring. The classes dual to the structure sheaves of Schubert varieties form the natural basis of $K^0(Fl_n)$. The class indexed by w is represented by the Grothendieck polynomial $\mathfrak{G}_w(x)$ (defined below). This is a nonhomogeneous polynomial in $x_1, \ldots, x_{n-1}$ whose lowest homogeneous component is the Schubert polynomial $\mathfrak{S}_w(x)$.

Note that both structure theorems above (for $H^*(Fl_n)$ and $K^0(Fl_n)$) hold, more generally, for flag bundles $Fl(V)$ corresponding to a vector bundle V of finite rank n on a variety X. In this case, $H^*(Fl(V))$ and $K^0(Fl(V))$ are free modules over $H^*(X)$ and $K^0(X)$ with bases given by Schubert and Grothendieck polynomials, respectively.

Both the Schubert polynomials $\mathfrak{S}_w(x)$ and the Grothendieck polynomials $\mathfrak{G}_w(x)$, for w in S_∞, form bases of $\mathbb{Z}[x_1, x_2, \ldots]$; here $S_\infty := \bigcup_n S_n$ under the usual inclusion $S_n \hookrightarrow S_{n+1}$.

The construction of Schubert and Grothendieck polynomials is based on the *divided difference operators* ∂_i and the *isobaric divided difference operators* π_i, which were originally introduced in the more general context of flag varieties corresponding to Kac-Moody Lie algebras. As operators on $P := \mathbb{Z}[x_1, x_2, \ldots]$, these are defined as follows:

$$\partial_i := \frac{1 - s_i}{x_i - x_{i+1}}\,, \quad \pi_i := \partial_i(1 - x_{i+1}) = 1 + (1 - x_i)\partial_i\,; \tag{2.1}$$

here s_i is the adjacent transposition $t_{i,i+1}$ acting in the obvious way on P, 1 is the identity on P, and x_i denotes multiplication by x_i. We will need the following formula for the action of π_i on a product of two polynomials, which is derived from a similar formula for ∂_i (see [18] p.23):

$$\pi_i(fg) = (\pi_i\, f)\, g + (s_i\, f)((\pi_i - 1)\, g)\,. \tag{2.2}$$

Schubert and Grothendieck polynomials are defined inductively for each w in S_n by setting $\mathfrak{S}_{w_0^n}(x) = \mathfrak{G}_{w_0^n}(x) = x_1^{n-1}x_2^{n-2}\ldots x_{n-1}$ for $w_0^n := [n, n-1, \ldots, 1]$ the longest permutation in S_n (we use the one-line notation for permutations throughout, unless otherwise specified), and by letting

$$\partial_i \, \mathfrak{S}_w(x) = \mathfrak{S}_{ws_i}(x)\,, \quad \pi_i \, \mathfrak{G}_w(x) = \mathfrak{G}_{ws_i}(x)\,, \quad \text{if } l(ws_i) = l(w) - 1\,. \tag{2.3}$$

The definition of Grothendieck polynomials does not depend on the chosen chain in the weak order on S_n from w_0^n to w because the operators π_i satisfy the braid relations

$$\begin{aligned} &\pi_i\pi_j = \pi_j\pi_i \quad \text{if } |i-j| \geq 2\,,\\ &\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}\,; \end{aligned}$$

similarly for Schubert polynomials. In addition, we have $\partial_i^2 = 0$ and $\pi_i^2 = \pi_i$. The latter relation implies the following counterpart to (2.3):

$$\pi_i \, \mathfrak{G}_w(x) = \mathfrak{G}_w(x) \quad \text{if } l(ws_i) = l(w) + 1\,. \tag{2.4}$$

Also note that Schubert and Grothendieck polynomials have a stability property, in the sense that their definition does not depend on n; this allows us to define them for w in S_∞.

Finally, we mention the classical Monk formula. But first let us recall the Bruhat order on the symmetric group, and its suborder known as the k-Bruhat order, where k is a positive integer. The Bruhat order was first defined and studied by Ehresmann [6]; it is given by its covering relations $w \leq wt_{ab}$, where $l(wt_{ab}) = l(w) + 1$. The k-Bruhat order first appeared in the context of Monk's formula, and was studied in more detail in [3]; it is given by the same covering relations as the Bruhat order, with the exception of the extra condition $a \leq k < b$. Note that both the Bruhat and the k-Bruhat orders have nonrecursive characterizations, which are not mentioned here. With these definitions, the classical Monk formula can be stated as

$$\mathfrak{S}_v(x)\,(x_1 + \ldots + x_k) = \mathfrak{S}_v(x)\,\mathfrak{S}_{s_k}(x) = \sum_w \mathfrak{S}_w(x)\,, \tag{2.5}$$

where the summation is over all covers w of v in the k-Bruhat order.

Let us also mention at this point the Pieri formula for Schubert polynomials, since it is interesting to compare the chains in k-Bruhat order it involves and those in the generalized Monk formula below. As mentioned in the introduction, these two formulas have a common generalization, which is the Pieri-type formula for Grothendieck polynomials in [17]. Recall that the Schubert polynomial indexed by the cycle $(k-p+1, k-p+2, \ldots, k+1)$ is the elementary symmetric polynomial $e_p(x_1, \ldots, x_k)$. We have

$$\mathfrak{S}_v(x)\,e_p(x_1, \ldots, x_k) = \sum_\gamma \mathfrak{S}_{\mathrm{end}(\gamma)}(x)\,, \tag{2.6}$$

where the summation is over all saturated chains in k-Bruhat order

$$\gamma := (v = w_0 <_k w_1 <_k \ldots <_k w_p = \mathrm{end}(\gamma))\,,$$

where $w_i = w_{i-1}t_{a_ib_i}$,

$$b_p \leq b_{p-1} \leq \ldots \leq b_1\,, \quad \text{and} \quad a_i \neq a_j \ \text{ for all } \ i \neq j\,.$$

It is not hard to see that this formula is multiplicity free.

3. Multiplying a Grothendieck polynomial by a single variable

Let v be a fixed permutation in S_∞, and k a positive integer. Define the set $P_k(v)$ to consist of all permutations

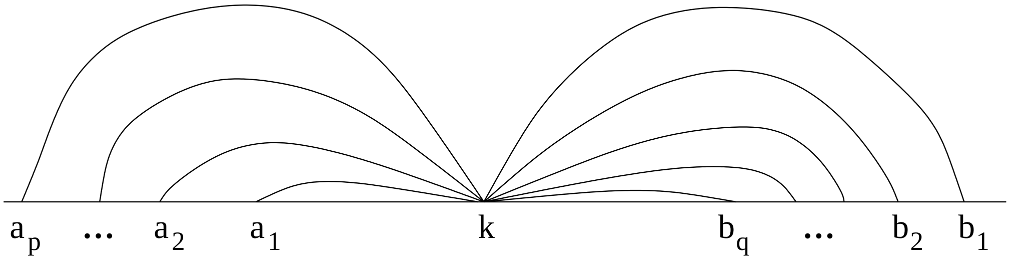
$$w = vt_{a_1k} \dots t_{a_pk}t_{kb_1} \dots t_{kb_q} ,$$

in S_∞ such that $p, q \geq 0$, $p+q \geq 1$, the length increases by precisely 1 upon multiplication by each transposition, and

$$a_p < a_{p-1} < \dots < a_1 < k < b_q < b_{q-1} < \dots < b_1 .$$

Given a permutation w in $P_k(v)$, define $\sigma_k(w, v) := (-1)^{q+1}$.

The order in which the transpositions above are applied is best seen in the picture below.



Theorem 3.1. *We have*

$$x_k\, \mathfrak{G}_v(x) = \sum_{w \in P_k(v)} \sigma_k(w, v)\mathfrak{G}_w(x) .$$

Proof. We may assume that v is a permutation in S_n and $k \leq n$. We will prove the above statement by decreasing induction on the length of v. If $v = w_0^n$ is the longest permutation in S_n, we have

$$x_k\, \mathfrak{G}_v(x) = \mathfrak{G}_{vt_{k,n+1}}(x) ;$$

indeed, in this case both v and $vt_{k,n+1}$ are *dominant* permutations, and it is known (see [12]) that

$$\mathfrak{G}_w(x) = x_1^{c_1} x_2^{c_2} \dots ,$$

for any dominant permutation w, where $(c_1, c_2, \dots)$ is the *code* of w (recall that a dominant permutation is one whose code is a weakly decreasing sequence).

It is now enough to show that if the formula holds for a permutation v in S_n (for any $k = 1, \dots, n$), it also holds for vs_i, where $1 \leq i < n$ and $l(vs_i) = l(v) - 1$. We fix i and consider the following cases, depending on the value of k.

Case 1. Assume that $k < i$. According to (2.2), we have

$$x_k\, \mathfrak{G}_{vs_i}(x) = \pi_i(x_k\mathfrak{G}_v(x)) . \tag{3.2}$$

Consider the following sets of permutations

$$P_k'(v) := \{ws_i \; : \; w \in P_k(v) \quad \text{and} \quad l(ws_i) = l(w) - 1\} ,$$
$$P_k''(v) := \{w \; : \; w \in P_k(v) \quad \text{and} \quad l(ws_i) = l(w) + 1\} .$$

Using induction, (3.2), (2.3), and (2.4), it follows that

$$x_k\, \mathfrak{S}_{vs_i}(x) = \sum_{w\in P'_k(v)} \sigma_k(ws_i,v)\mathfrak{S}_w(x) + \sum_{w\in P''_k(v)} \sigma_k(w,v)\mathfrak{S}_w(x)\,. \tag{3.3}$$

Hence we need to show that

$$\sum_{w\in P'_k(v)} \sigma_k(ws_i,v)\mathfrak{S}_w(x) + \sum_{w\in P''_k(v)} \sigma_k(w,v)\mathfrak{S}_w(x) = \sum_{w\in P_k(vs_i)} \sigma_k(w,vs_i)\mathfrak{S}_w(x)\,. \tag{3.4}$$

Note that if we restrict formula (3.4) to the permutations in $P'_k(v)$ and $P_k(vs_i)$ of length $l(v)$, we obtain essentially Monk's formula for Schubert polynomials. For simplicity we restrict (3.4) to the permutations in $P'_k(v)$, $P''_k(v)$, and $P_k(vs_i)$ of length $l(v)+1$, since the cases of greater lengths are similar. Hence we consider permutations of the form $vt_{kl_1}t_{kl_2}s_i$ in $P'_k(v)$, of the form vt_{kl} in $P''_k(v)$, and of the form $vs_it_{kl_1}t_{kl_2}$ in $P_k(vs_i)$. The restriction of formula (3.4) to these permutations follows from the following remarks.

1. Assuming $\{l_1,l_2\}\cap\{i,i+1\}=\emptyset$, we have that $w=vt_{kl_1}t_{kl_2}s_i$ lies in $P'_k(v)$ if and only if $w=vs_it_{kl_1}t_{kl_2}$ lies in $P_k(vs_i)$, and $\sigma_k(ws_i,v)=\sigma_k(w,vs_i)$.
2. We have that $w=vt_{ki}s_i$ lies in $P'_k(v)$ if and only if $w=vs_it_{k,i+1}$ lies in $P_k(vs_i)$, and $\sigma_k(ws_i,v)=\sigma_k(w,vs_i)$. On the other hand, $w=vt_{k,i+1}s_i$ lies in $P'_k(v)$ if and only if $w=vs_it_{ki}$ lies in $P_k(vs_i)$, and $\sigma_k(ws_i,v)=\sigma_k(w,vs_i)$.
3. Based on the previous remark, it follows that $w=vt_{kl_1}t_{kl_2}s_i$ lies in $P'_k(v)$, where the sets $\{l_1,l_2\}$ and $\{i,i+1\}$ have precisely one element in common, if and only if w lies in $P_k(vs_i)$, and $\sigma_k(ws_i,v)=\sigma_k(w,vs_i)$.
4. We have that $w=vt_{ki}$ lies in $P''_k(v)$ if and only if $w=vt_{k,i+1}t_{ki}s_i$ lies in $P'_k(v)$, and $\sigma_k(w,v)=1$, $\sigma_k(ws_i,v)=-1$; hence the two terms corresponding to these permutations on the left-hand side of (3.4) cancel out.
5. The permutation vt_{kl} does not lie in $P''_k(v)$ for $l\neq i$, and the permutation $vs_it_{k,i+1}t_{ki}$ does not lie in $P_k(vs_i)$.

Case 2. Assume that $k>i+1$. Formula (3.3) still holds, so we still need to prove (3.4). We proceed in a similar way. Remarks 1), 2), and 3) still hold, while remarks 4) and 5) are replaced with the following ones.

4'. We have that $w=vt_{k,i+1}$ lies in $P''_k(v)$ if and only if $w=vs_it_{k,i+1}t_{ki}$ lies in $P_k(vs_i)$, and $\sigma_k(w,v)=\sigma_k(w,vs_i)=-1$; hence we have two equal terms on the left and right-hand sides of (3.4).

5'. The permutation vt_{kl} does not lie in $P''_k(v)$ for $l\neq i+1$, and the permutation $vt_{k,i+1}t_{ki}s_i$ does not lie in $P'_k(v)$.

Case 3. Assume that $k=i$. According to (2.2), we have

$$x_i\, \mathfrak{S}_{vs_i}(x) = \pi_i(x_{i+1}\mathfrak{S}_v(x)) - x_{i+1}\mathfrak{S}_v(x) + \mathfrak{S}_v(x)\,. \tag{3.5}$$

Proceeding in a similar way to the previous cases, the induction step amounts to proving that

$$\begin{aligned}\sum_{w\in P'_{i+1}(v)} \sigma_{i+1}(ws_i,v)\mathfrak{S}_w(x) &- \sum_{w\in P_{i+1}(v)\setminus P''_{i+1}(v)} \sigma_{i+1}(w,v)\mathfrak{S}_w(x)\\ &+ \mathfrak{S}_v(x) = \sum_{w\in P_i(vs_i)} \sigma_i(w,vs_i)\mathfrak{S}_w(x)\,.\end{aligned} \tag{3.6}$$

Again, this is essentially Monk's formula when restricted to permutations of length $l(v)$. The formula restricted to permutations of length $l(v)+r$ for $r \geq 1$ follows from the following remarks.

1. We have that $w = vt_{i+1,l_1} \dots t_{i+1,l_{r+1}} s_i$ lies in $P'_{i+1}(v)$ if and only if $w = vs_i t_{il_1} \dots t_{il_{r+1}}$ lies in $P_i(vs_i)$, and $\sigma_{i+1}(ws_i, v) = \sigma_i(w, vs_i)$. Hence we have two equal terms on the two sides of (3.6). Note that in this case none of the indices $l_1, \dots, l_{r+1}$ equals i or $i+1$.
2. We have that $w = vt_{i+1,l_1} \dots t_{i+1,l_r}$ lies in $P_{i+1}(v) \setminus P''_{i+1}(v)$ if and only if $w = vs_i t_{il_1} \dots t_{il_r} t_{i,i+1}$ lies in $P_i(vs_i)$, and $\sigma_{i+1}(w,v) = -\sigma_i(w, vs_i)$. Hence we have two equal terms on the two sides of (3.6).

Case 4. Finally, assume that $k = i+1$. According to (2.2), we have

$$x_{i+1}\, \mathfrak{G}_{vs_i}(x) = \pi_i(x_i \mathfrak{G}_v(x)) + x_{i+1}\mathfrak{G}_v(x) - \mathfrak{G}_v(x)\,. \tag{3.7}$$

Proceeding in a similar way to the previous cases, the induction step amounts to proving that

$$\begin{aligned} \sum_{w \in P'_i(v)} \sigma_i(ws_i, v)\mathfrak{G}_w(x) &+ \sum_{w \in P''_i(v)} \sigma_i(w,v)\mathfrak{G}_w(x) \\ &+ \sum_{w \in P_{i+1}(v)} \sigma_{i+1}(v,w)\mathfrak{G}_w(x) - \mathfrak{G}_v(x) = \sum_{w \in P_{i+1}(vs_i)} \sigma_{i+1}(w, vs_i)\mathfrak{G}_w(x)\,. \end{aligned} \tag{3.8}$$

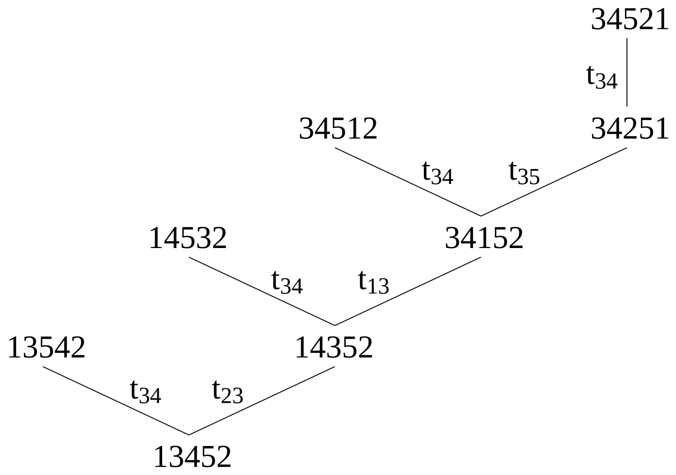
Again, this is essentially Monk's formula when restricted to permutations of length $l(v)$. The formula restricted to permutations of length $l(v)+r$ for $r \geq 1$ follows from the following remarks.

1. We have that $w = vt_{il_1} \dots t_{il_{r+1}} s_i$ lies in $P'_i(v)$ and $l_{r+1} \neq i+1$ if and only if $w = vs_i t_{i+1,l_1} \dots t_{i+1,l_{r+1}}$ lies in $P_{i+1}(vs_i)$; furthermore, $\sigma_i(ws_i, v) = \sigma_{i+1}(w, vs_i)$. Hence we have two equal terms on the two sides of (3.8). Note that in this case none of the indices $l_1, \dots, l_{r+1}$ equals i or $i+1$.
2. We have that $w = vt_{il_1} \dots t_{il_r} t_{i,i+1} s_i$ lies in $P'_i(v)$ if and only if $w = vt_{il_1} \dots t_{il_r}$ lies in $P''_i(v)$, and $\sigma_i(ws_i, v) = -\sigma_i(w,v)$. Hence the corresponding two terms on the left-hand side of (3.8) cancel out.
3. We have that $w = vt_{i+1,l_1} \dots t_{i+1,l_r}$ lies in $P_{i+1}(v)$ if and only if $w = vs_i t_{i+1,i} t_{i+1,l_1} \dots t_{i+1,l_r}$ lies in $P_{i+1}(vs_i)$; furthermore, $\sigma_{i+1}(w,v) = \sigma_{i+1}(w, vs_i)$. Hence we have two equal terms on the two sides of (3.8). □

Example 3.9. We have

$$\begin{aligned} x_3\, \mathfrak{G}_{13452}(x) = (\mathfrak{G}_{13542}(x) - \mathfrak{G}_{14352}(x)) + (\mathfrak{G}_{14532}(x) - \mathfrak{G}_{34152}(x)) + \\ + (\mathfrak{G}_{34512}(x) + \mathfrak{G}_{34251}(x)) - \mathfrak{G}_{34521}(x)\,. \end{aligned}$$

The brackets contain the Grothendieck polynomials indexed by permutations of the same length, which have the same degrees of their lowest homogeneous components. The following tree illustrates the way in which the permutations in the formula are constructed; the edges are labeled by the corresponding transpositions.



We easily deduce, as a corollary to Theorem 3.1, the transition formula for Grothendieck polynomials, which forms the object of [13]. Recall that if k is the last descent of the permutation v in S_∞ (that is, the greatest k such that $v(k) > v(k+1)$), then $\mathfrak{G}_v(x)$ is a polynomial in $x_1, \ldots, x_k$.

Corollary 3.10. [13] *Let k be the last descent of v, and let $u := vt_{kl}$, where l is defined by*

$$v(l) = \max \{v(i) \,:\, i > k \quad \textit{and} \quad v(i) < v(k)\}\,.$$

Let $1 \le i_1 < \ldots < i_m < k$ be the positions for which $l(ut_{i_j k}) = l(u) + 1$. Then

$$\mathfrak{G}_v(x) = \mathfrak{G}_u(x) + (x_k - 1)\left[\mathfrak{G}_u(x)\cdot(1 - t_{i_1 k}) \ldots (1 - t_{i_m k})\right]\,,$$

where t_{jk} acts on $\mathfrak{G}_w(x)$ by $\mathfrak{G}_w(x)\cdot t_{jk} := \mathfrak{G}_{wt_{jk}}(x)$.

Proof. Let $I := \{i_1, \ldots, i_m\}$, and for any $J \subseteq I$, let

$$t_{Jk} := t_{j_1 k} \ldots t_{j_p k}\,,$$

where $J := \{j_1 < \ldots < j_p\}$. By Theorem 3.1, the formula to prove is equivalent to

$$\sum_{\emptyset \ne J \subseteq I} (-1)^{|J|+1} \mathfrak{G}_{ut_{Jk}}(x) + \sum_{J \subseteq I} \sum_{w \in P_k(ut_{Jk})} (-1)^{|J|} \sigma_k(w, ut_{Jk}) \mathfrak{G}_w(x) = \mathfrak{G}_v(x)\,. \tag{3.11}$$

Each permutation w indexing a Grothendieck polynomial on the left-hand side of (3.11) can be written as

$$w = ut_{j_1 k} \ldots t_{j_p k} t_{j_{p+1} k} \ldots t_{j_q k} t_{j_{q+1} k} \ldots t_{j_r k}\,,$$

where

$$j_1 < \ldots < j_{p-1} < j_p > j_{p+1} > \ldots > j_q\,, \quad j_{q+1} > \ldots > j_r > k\,, \quad \text{and} \quad \{j_1, \ldots, j_p\} \subseteq I\,.$$

Assuming that $q > 0$, the permutation w appears precisely twice on the left-hand side of (3.11); indeed, the two appearances correspond to $J = \{j_1, \ldots, j_p\}$ and $J = \{j_1, \ldots, j_{p-1}\}$. Furthermore, the corresponding two signs are opposite. On the other hand, it is easy to check that there is a unique permutation with $q = 0$, namely v. Hence (3.11) holds. $\square$

4. Monk's formula for Grothendieck polynomials

Let v be again a fixed permutation in S_∞, and k a fixed positive integer. Define the following order on pairs of positive integers:

$$(a,b) \leq (c,d) \quad \text{if and only if} \quad (b > d) \quad \text{or} \quad (b = d \quad \text{and} \quad a \leq c)\,. \tag{4.1}$$

Consider the set $\Gamma_k(v)$ consisting of saturated chains in the k-Bruhat order

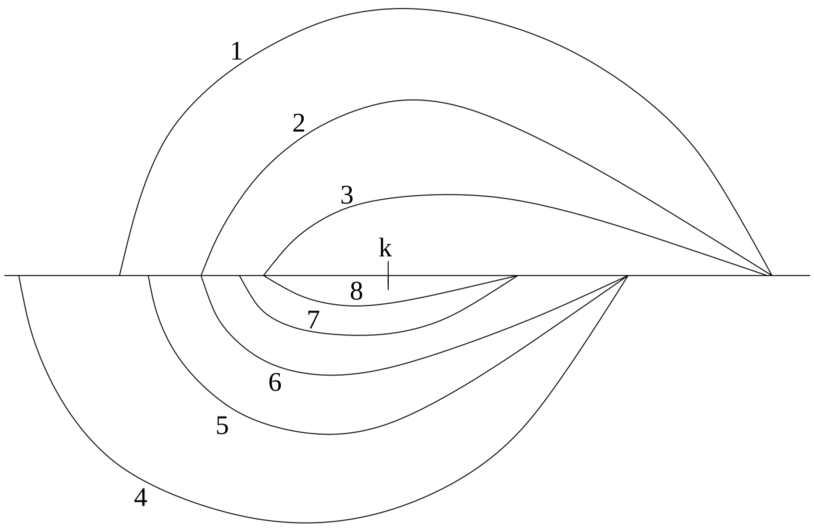
$$v = w_0 <_k w_1 <_k \ldots <_k w_p\,,$$

where $p \geq 1$, $w_i = w_{i-1} t_{a_i b_i}$ (with $a_i < b_i$), and

$$(a_1, b_1) < (a_2, b_2) < \ldots < (a_p, b_p)\,. \tag{4.2}$$

Given any chain γ, we denote by $\mathrm{end}(\gamma)$ its last element, and by $l(\gamma)$ its length. Furthemore, we define the set $\Pi_k(v)$ to consist of all permutations $\mathrm{end}(\gamma)$ for γ in $\Gamma_k(v)$.

The order in which the transpositions above are applied is best seen in the picture below, being indicated by the numbers on the corresponding curves. It is interesting to notice, by comparison, that these chains in k-Bruhat order differ from those in the Pieri formula for Schubert polynomials (2.6) in two ways: (1) there might be coincidences between the left positions a_i in the corresponding transpositions; (2) the left positions a_i of the successive transpositions having the same right position b_j form an increasing sequence, whereas in (2.6) they are unordered. Being a common generalization of (2.6) and Theorem 4.3 below, the Pieri-type formula for Grothendieck polynomials in [17] involves chains in k-Bruhat order which generalize the two types of chains mentioned above.



Theorem 4.3. *We have*

$$\mathfrak{G}_v(x)\,\mathfrak{G}_{s_k}(x) = \sum_{w \in \Pi_k(v)} (-1)^{l(w)-l(v)+1} \mathfrak{G}_w(x)\,.$$

Proof. We first prove the closely related formula

$$\mathfrak{G}_v(x)\,\mathfrak{G}_{s_k}(x) = \sum_{\gamma \in \Gamma_k(v)} (-1)^{l(\gamma)+1} \mathfrak{G}_{\mathrm{end}(\gamma)}(x)\,. \tag{4.4}$$

It is known from [12], as well as [7, 16], that

$$\mathfrak{G}_{s_k}(x) = 1 - (1 - x_1) \dots (1 - x_k)\,, \tag{4.5}$$

whence

$$\mathfrak{G}_{s_k}(x) = \mathfrak{G}_{s_{k-1}}(x) + x_k - \mathfrak{G}_{s_{k-1}}(x)x_k\,. \tag{4.6}$$

Note that the latter formula is just a special case of the transition formula in Corollary 3.10. Thus, we can prove (4.4) by induction on k. For $k = 1$ it is just a special case of Theorem 3.1, since $\mathfrak{G}_{s_1}(x) = x_1$.

Each permutation w in the set $P_k(v)$ (see the definition at the beginning of the previous section) determines a unique saturated chain in Bruhat order from v to w. Let $C_k(v)$ denote the collection of these chains. Let $\widehat{C}_k(v)$ and $\widehat{\Gamma}_k(v)$ denote the sets obtained from $C_k(v)$ and $\Gamma_k(v)$ by adding the chain of length 0 consisting of the permutation v only. Given a chain γ in $C_k(v)$, we denote by $\widetilde{l}(\gamma)$ the number of permutations in γ obtained by applying a transposition t_{kl} with $l > k$ to the previous permutation.

Assuming that (4.4) holds for $k - 1$, and using (4.6), we have

$$\mathfrak{G}_v(x)\,\mathfrak{G}_{s_k}(x) = \sum_{(\gamma_1,\gamma_2)} (-1)^{l(\gamma_1)+\widetilde{l}(\gamma_2)+1}\,\mathfrak{G}_{\mathrm{end}(\gamma_2)}(x)\,; \tag{4.7}$$

here the summation ranges over all pairs (γ_1, γ_2) of chains, not both of length 0, with γ_1 in $\widehat{\Gamma}_{k-1}(v)$ and γ_2 in $\widehat{C}_k(\mathrm{end}(\gamma_1))$.

We now define a sign-reversing involution on the above set of pairs (γ_1, γ_2). Let $\gamma_1 := (v = w_0 < w_1 < \dots < w_p)$ and $\gamma_2 := (w_p = w_0' < w_1' < \dots < w_q')$; let $w_i := w_{i-1}t_{a_ib_i}$ and $w_j' := w_{j-1}'t_{kl_j}$. It is convenient to represent the two chains by the sequences $(t_{a_1b_1}, \dots, t_{a_pb_p})$ and $(t_{kl_1}, \dots, t_{kl_q})$. If none of the two sequences contains a transposition of the form t_{lk} with $l < k$, then the pair (γ_1, γ_2) is fixed by the involution. Assume that $p > 0$, and $b_p = k$, and at least one of the following holds

$$q = 0\,, \quad \text{or} \quad l_1 > k\,, \quad \text{or} \quad a_p > l_1\,.$$

Then we map the pair (γ_1, γ_2) to the pair of chains represented by the sequences of tranpositions

$$(t_{a_1b_1}, \dots, t_{a_{p-1}b_{p-1}}) \quad \text{and} \quad (t_{a_pk}, t_{kl_1}, \dots, t_{kl_q})\,.$$

In the remaining case, we have $p = 0$, or $b_p \neq k$, or the following (simultaneously):

$$q > 0\,, \quad \text{and} \quad b_p = k\,, \quad \text{and} \quad a_p < l_1 < k\,.$$

Then we map the pair (γ_1, γ_2) to the pair represented by

$$(t_{a_1b_1}, \dots, t_{a_pb_p}, t_{l_1k}) \quad \text{and} \quad (t_{kl_2}, \dots, t_{kl_q})\,.$$

It is not difficult to check that the map defined above is indeed an involution on the set over which the summation in (4.7) ranges. Furthemore, the involution is clearly sign-reversing, because it increases or decreases $l(\gamma_1)$ by 1, but does not change $\widetilde{l}(\gamma_2)$. Hence, all terms in the summation in (4.7), except those corresponding to the fixed points of the involution, cancel; indeed, the involution preserves $\mathrm{end}(\gamma_2)$. Thus we proved that (4.4) holds.

In order to deduce Theorem 4.3 from (4.4), we need to show that the summation in the latter formula is multiplicity free (after collecting terms). This is done in the following Proposition, and thus the proof is complete. □

Proposition 4.8. *For any w in $\Pi_k(v)$, there is a unique chain γ in $\Gamma_k(v)$ with* $\mathrm{end}(\gamma) = w$.

Proof. We assume that v, w lie in S_n. Let γ be a chain in $\Gamma_k(v)$ with $\mathrm{end}(\gamma) = w$, which we represent as a sequence of transpositions $t := (t_{a_1b_1}, \ldots, t_{a_1b_r}, t_{a_2c_1}, \ldots, t_{a_2c_s}, \ldots)$, as we did in the proof of Theorem 4.3; we have $a_1 < a_2 < \ldots$, $b_1 > \ldots > b_r$, $c_1 > \ldots > c_s$ etc. First note that $v(i) = w(i)$ implies that none of the transpositions above act on position i. Hence we may assume without loss of generality that $v(i) \neq w(i)$ for all $i = 1, \ldots, n$; thus $a_1 = 1$, a_2=2 etc.

Position b_r is clearly determined by v and w only; in fact $b_r = v^{-1}(w(1))$. Let $\alpha := v(1)$ and $\beta := v(b_r) = w(1)$. We will show that $b_1, \ldots, b_{r-1}$ are also determined by v and w. We claim that we have either $v(i) < \alpha$ or $v(i) > \beta$ for all i satisfying $1 < i < b_r$ or $b_j < i < b_{j-1}$ for some $j = 1, \ldots, r$, where we set $b_0 := n$. This implies that $r = \beta - \alpha$ and $b_j = v^{-1}(\alpha + j)$ for $j = 1, \ldots, \beta - \alpha - 1$. The uniqueness of $c_1, \ldots, c_s, \ldots$ follows by an inductive argument.

The above claim is clearly true for i with $1 < i < b_r$; indeed, otherwise one of the transpositions $t_{1b_1}, \ldots, t_{1b_r}$ would increase length by more than 1 when successively multiplying v on the right (recall that $l(wt_{ab}) = l(w) + 1$ if and only if $w(a) < w(b)$ and either $w(i) < w(a)$ or $w(i) > w(b)$ for all $i = a + 1, \ldots, b - 1$). Now consider i with $b_j < i < b_{j-1}$ for some $j = 1, \ldots, r$. Let t_{mi} be the leftmost transposition in the sequence t which acts on position i, and let v' be the permutation in the chain γ it multiplies on the right. We may assume that $v(m) < \alpha$, since otherwise $v(m) > \beta$, whence $v(i) > \beta$. Let $\gamma := v(b_{j-1})$ if $j > 1$, and $\gamma := \alpha$ otherwise. By an argument similar to one used above, we have that $v(i) < \alpha$ or $v(i) > \gamma$. Hence, it is enough to show that we cannot have $v(i) = v'(i) > \gamma$. Assuming that this is true, we are led to a contradiction once we prove that $v(m) < v'(b_j) \leq \gamma$, since it would follow that the transposition t_{mi} or one preceding it (and still acting on position m) increases length by more than 1.

The only thing left to prove is that $v(m) < v'(b_j) \leq \gamma$. The second inequality is clear since $(vt_{1b_1} \ldots t_{1b_r})(b_j) = \gamma$, and the value on position b_j weakly decreases upon multiplication by subsequent transpositions. Now assume that $v'(b_j) < v(m)$. Then there has to be a permutation v'' in the chain γ which is multiplied on the right by a transposition t_{lb_j}, such that $l < m$, $v''(l) < v(m)$, and $v''(b_j) > v(m)$; indeed, $(vt_{1b_1} \ldots t_{1b_r})(b_j) = \gamma \geq \alpha > v(m)$, and the value on position b_j weakly decreases. But then the length increases by more than 1 upon multiplying v'' by t_{lb_j}, which is not possible. □

The above proof provides the following algorithm for finding the unique chain in k-Bruhat order from a permutation v to a permutation w in $\Pi_k(v)$. More explicitly, we want to find the sets B_i (possibly empty) for $i = 1, \ldots, k$, such that

$$w = v \overrightarrow{\prod_{i \in \{1,\ldots,k\}}} \overleftarrow{\prod_{b \in B_i}} t_{ib} \, ,$$

where the right and left arrows indicate that the indices in the noncommutative products are increasing and decreasing, respectively.

Algorithm 4.9.

Step 1. *Let* $i := 1$.

Step 2. *If* $v(i) = w(i)$ *then* $B_i := \emptyset$*; go to Step 5.*

Step 3. *Let*

$$B_i := \{j \, : \, j > k \, , \;\; v(j) \neq w(j) \, , \quad \textit{and} \quad v(i) < v(j) \leq w(i)\} \, .$$

Step 4. *Let*

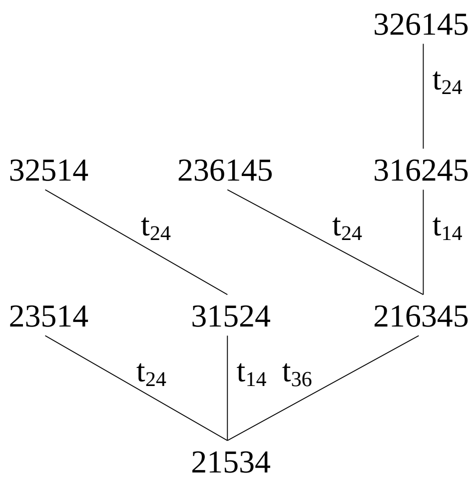
$$v := v \overleftarrow{\prod_{b \in B_i}} t_{ib} \, .$$

Step 5. *Let* $i := i + 1$*; if* $v \neq w$ *then go to Step 2 else output the sets* B_i*. STOP.*

Example 4.10. We have

$$\begin{aligned} \mathfrak{G}_{21534}(x)\, \mathfrak{G}_{s_3}(x) = (&\mathfrak{G}_{23514}(x) + \mathfrak{G}_{31524}(x) + \mathfrak{G}_{216345}(x)) - \\ &- (\mathfrak{G}_{32514}(x) + \mathfrak{G}_{236145}(x) + \mathfrak{G}_{316245}(x)) + \mathfrak{G}_{326145}(x) \, . \end{aligned}$$

The brackets contain the Grothendieck polynomials indexed by permutations of the same length, which have the same degrees of their lowest homogeneous components. The following tree illustrates the way in which the permutations in the formula are constructed; the edges are labeled by the corresponding transpositions.



It would be interesting to compare the algorithmic complexity of generating these trees of permutations with the complexity of applying the operators in [12]. Recall that these operators give the expansion of the product in the Monk-type formula with cancellations, whereas our formula has no cancellations. So one would expect the latter to be more efficient.

It was shown in [12] that if the permutation v in S_n has unique ascent at some position k, that is,

$$v(1) > v(2) > \ldots > v(k) < v(k+1) > v(k+2) > \ldots > v(n) \, ,$$

then $\Pi_k(v)$ is the whole interval $(v, w_0^n]$ in Bruhat order (recall that $w_0^n := [n, n-1, \ldots, 1]$). We can rederive this result from Theorem 4.3, but, since it seems that the proof is slightly longer than the one in [12], we do not reproduce it here. Instead, we rederive the special case of the Pieri-type formula in [16] for *Grassmannian permutations*, which corresponds to multiplication by $\mathfrak{G}_{s_k}(x)$.

Recall that a permutation is called Grassmannian if it has a unique descent at some position k, that is,

$$v(1) < v(2) < \ldots < v(k) > v(k+1) < v(k+2) < \ldots .$$

The name is motivated by the fact that the corresponding Schubert classes (both in cohomology and K-theory) are pulled back from a Grassmannian projection. Grassmannian permutations v are best represented by their corresponding partition

$$\lambda(v) = (v(k) - k \geq v(k-1) - k + 1 \geq \ldots \geq v(1) - 1 \geq 0)\,.$$

Conversely, given k and a partition λ with at most k parts, there is a unique Grassmannian permutation $v(\lambda, k)$ with descent at k corresponding to the partition λ. It turns out that the Grothendieck polynomial $\mathfrak{G}_{v(\lambda,k)}(x)$ is a symmetric polynomial in $x_1, \ldots, x_k$; thus, it will be denoted by $\mathfrak{G}_\lambda(x_1, \ldots, x_k)$. In particular, $\mathfrak{G}_{(1)}(x_1, \ldots, x_k) = \mathfrak{G}_{s_k}(x)$. With this notation, a special case of Theorems 3.1 and 3.2 in [16] can be restated as follows.

Corollary 4.11. [16] *We have*

$$\mathfrak{G}_\lambda(x_1, \ldots, x_k)\, \mathfrak{G}_{(1)}(x_1, \ldots, x_k) = \sum_{\mu} (-1)^{|\mu|-|\lambda|} \mathfrak{G}_\mu(x_1, \ldots, x_k)\,.$$

Here $|\mu|, |\lambda|$ denote, as usual, the sum of parts of the corresponding partitions; the summation ranges over all partitions μ with at most k parts whose Young diagrams are obtained from that of λ by adding $|\mu| - |\lambda|$ boxes, with no two in the same row or column.

Proof. The fact that v is Grassmannian with descent at k combined with condition (4.2) on the saturated chains γ in k-Bruhat order appearing in Theorem 4.3 implies that

$$a_p < a_{p-1} < \ldots < a_1\,, \quad b_p < b_{p-1} < \ldots < b_1 \;\text{ and }\; v(b_i) - v(a_i) = 1\,,\; i = 1, \ldots, p\,.$$

This translates into the above condition on the partitions μ corresponding to $\mathrm{end}(\gamma)$; clearly, $\mathrm{end}(\gamma)$ is still Grassmannian with descent at k. $\square$

Remark 4.12. Theorem 4.3 still holds if we use a slightly different order on pairs on positive integers than the one defined in (4.1), namely

$$(a, b) \preceq (c, d) \;\text{ if and only if }\; (a < c) \;\text{ or }\; (a = c \;\text{ and }\; b \geq d)\,; \tag{4.13}$$

all the other definitions remain unchanged. This fact is deduced from the following observation: if

$$(a, b) \leq (c, d) \quad\text{and}\quad (a, b) \succeq (c, d)\,,$$

then the integers a, b, c, d are all distinct, whence the transpositions t_{ab} and t_{cd} commute. The same is true if

$$(a, b) \preceq (c, d) \quad\text{and}\quad (a, b) \geq (c, d)\,.$$

In fact, it turns out that the two orders $\leq$ and $\preceq$ on pairs are used in the Pieri-type formulas in [17] for multiplying $\mathfrak{G}_v(x)$ by $\mathfrak{G}_{(1^p)}(x_1, \ldots, x_k)$ and $\mathfrak{G}_{(p)}(x_1, \ldots, x_k)$, respectively. This is the reason for which both work in the context here.

5. Pieri operators and related multiplication formulas

In this section we introduce two new families of noncommutative Pieri operators on the Bruhat order and the k-Bruhat order on the symmetric group, which are inspired by Theorems 3.1 and 4.3. Pieri operators on posets were introduced in [2]. We briefly recall the main elements of the constructions involved.

Let $(P, \leq)$ be a graded poset with rank function $rk : P \to \mathbb{Z}_+$, and let $\mathbb{Z}P$ be the free graded $\mathbb{Z}$-module generated by the elements of P. For an integer $m > 0$, a (right) Pieri operator on P is a linear map $\overline{h}_m : \mathbb{Z}P \to \mathbb{Z}P$ which respects the poset structure; by this it is meant that for all x in P, the support of $x \cdot \overline{h}_m$ in $\mathbb{Z}P$ consists only of elements y in P such that $x < y$ and $rk(y) - rk(x) = m$.

Now recall the Hopf algebra NC of noncommutative symmetric functions defined in [10]; as an algebra, this is just the free associative algebra over $\mathbb{Z}$ with a generator h_m in each positive degree m. Its dual is the Hopf algebra $\mathcal{Q}sym$ of quasi-symmetric functions (see [11, 10]). The latter has two remarkable bases, namely the *monomial quasi-symmetric functions* M_α and the *fundamental quasi-symmetric functions* F_α; both are indexed by *compositions* $\alpha = (\alpha_1, \ldots, \alpha_l)$ of positive integers of arbitrary *length* $l = l(\alpha)$. Note that the basis $\{M_\alpha\}$ is dual to the basis of *complete NC-functions* $\{S^\alpha := h_{\alpha_1} \ldots h_{\alpha_l}\}$.

Given an infinite family of Pieri operators $\{\overline{h}_m \,:\, m = 1, 2, \ldots\}$, the map $h_m \mapsto \overline{h}_m$ turns $\mathbb{Z}P$ into a (right) graded NC-module. Given x, y in P, the association to any Ψ in NC of the coefficient of y in $x \cdot \Psi$ is a linear map on NC. Hence, these matrix coefficients are elements of $\mathcal{Q}sym$, and we have a collection of quasi-symmetric functions $K_{[x,y]}$ associated to every interval $[x, y]$ of P.

Now let $\mathcal{H}P$ be the free $\mathbb{Z}$-module with basis given by cartesian products of intervals of P, modulo identifying all singleton intervals $[x, x]$ with the unit 1, and empty intervals with 0. Then $\mathcal{H}P$ is a graded $\mathbb{Z}$-algebra whose product is cartesian product of intervals, and there is a natural coalgebra structure which turns it into a Hopf algebra (see [22]). According to [2], the map which associates to every interval $[x, y]$ of P the quasi-symmetric function $K_{[x,y]}$ can be extended to a map K on $\mathcal{H}P$, and we have the following result justifying the construction.

Theorem 5.1. [2] *The map $K : \mathcal{H}P \to \mathcal{Q}sym$ is a morphism of Hopf algebras.*

An important family of Pieri operators, called *descent Pieri operators* in [2], are associated with (edge)-labeled posets. These are graded posets whose covers are labeled with elements of a totally ordered set; in fact, the definition in [2] is based on posets with edges labeled with integers, but for our purposes it is helpful to allow labelings with elements of any totally ordered set. One of the main examples of descent Pieri operators in [2] was suggested by the Pieri formula for Schubert polynomials (2.6). Since these Pieri operators commute, the corresponding quasi-symmetric functions are, in fact, symmetric, and contain information about the Littlewood-Richardson coefficients corresponding to the multiplication of a Schubert polynomial by a Schur polynomial.

Recall that given a labeled poset P, the descent Pieri operator is defined by

$$x\cdot\overline{h}_m := \sum_{\gamma} \mathrm{end}(\gamma)\,,$$

where the summation ranges over all saturated chains γ of length m starting at x,

$$\gamma \;:\; x \xrightarrow{\lambda_1} x_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_m} x_m = \mathrm{end}(\gamma)\,,$$

with no descents, that is, $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_m$. We define Pieri operators $\overline{e}_m$ similarly, with the only exception that we require (strictly) decreasing chains, that is, $\lambda_1 > \lambda_2 > \ldots > \lambda_m$. The reason for the notation is that, according to Proposition 5.4 below, the map $h_m \mapsto \overline{h}_m$ turning $\mathbb{Z}P$ into a graded NC-module maps the noncommutative elementary symmetric functions e_m to $\overline{e}_m$ (cf. the definition of e_m in [10]).

To a subset $I = \{i_1 < \ldots < i_p\}$ of $\{1, \ldots, m-1\}$, we associate the composition $\alpha(I) := (i_1, i_2 - i_1, \ldots, m - i_p)$ of m. Given a saturated chain γ in a labeled poset with labels $\lambda_1, \ldots, \lambda_m$, let $D(\gamma)$ be the *descent composition* of γ, that is, the composition associated to the descent set $\{i : \lambda_i > \lambda_{i+1}\}$ of γ. Then it was proved in [4] that the quasi-symmetric functions associated to descent Pieri operators can be written

$$K_{[x,y]} = \sum_{\gamma} F_{D(\gamma)}\,, \tag{5.2}$$

where the sum is over all saturated chains γ in the interval $[x, y]$.

We have the following result, for which we present two proofs: the first is more algebraic, while the second is more combinatorial. Let

$$H := \sum_{m=1}^{\infty} (-1)^{m-1}\overline{h}_m\,. \tag{5.3}$$

Proposition 5.4. *We have*

$$(1-H)^{-1} = \sum_{m=0}^{\infty} \overline{e}_m\,.$$

First proof. Consider $x < y$ in P with $rk(y) - rk(x) = m$. The coefficient of y in $x\cdot H^p$ multiplied by $(-1)^{m-p}$ is obtained by summing over all compositions $\alpha = (\alpha_1, \ldots, \alpha_p)$ of m the coefficients of y in $x\cdot\overline{h}_{\alpha_1}\ldots\overline{h}_{\alpha_p}$. But these are just the coefficients of M_α in $K_{[x,y]}$, by the definition of the latter. We deduce that the coefficient of y in $x\cdot(1-H)^{-1}$ is $(-1)^m f(K_{[v,w]})$, where $f : \mathcal{Q}sym \to \mathbb{Z}$ is the specialization map defined by $f(M_\alpha) = (-1)^{l(\alpha)}$. It is convenient to switch to the basis of fundamental quasi-symmetric functions. Recall that given a subset I of $[n-1] := \{1, \ldots, n-1\}$, we have

$$F_{\alpha(I)} = \sum_{I\subseteq J\subseteq[n-1]} M_{\alpha(J)}\,.$$

Thus, we have

$$f(F_{\alpha(I)}) = \begin{cases} (-1)^n & \text{if } I = [n-1] \\ 0 & \text{otherwise}\,. \end{cases}$$

Therefore, the coefficient of y in $x\cdot(1-H)^{-1}$ is just the coefficient of $F_{\alpha([m])}$ in the expansion of $K_{[x,y]}$ in the basis of fundamental quasi-symmetric functions. By (5.2), the latter coefficient is the number of decreasing saturated chains from x to y. $\square$

Second proof. We prove that the operator

$$\left(\sum_{m=0}^{\infty}(-1)^m\overline{h}_m\right)\left(\sum_{m=0}^{\infty}\overline{e}_m\right)$$

is just the identity. Thus, we have to prove that for $m > 0$ we have

$$\sum_{i=0}^{m}(-1)^i\overline{h}_i\overline{e}_{m-i} = 0\,.$$

In order to do this, we construct a sign-reversing involution on the saturated chains of length m determined by the operator on the left-hand side of the above identity. A typical chain γ produced by the operator $(-1)^i\overline{h}_i\overline{e}_{m-i}$ has two sequences of labels

$$(\lambda_1 \leq \ldots \leq \lambda_i) \quad \text{and} \quad (\lambda_{i+1} > \ldots > \lambda_m)\,,$$

where at most one can be empty. If we have $(i = m)$ or $(0 < i < m$ and $\lambda_i > \lambda_{i+1})$, then the same chain is produced by the operator $(-1)^{i-1}\overline{h}_{i-1}\overline{e}_{m-i+1}$. Otherwise, we have $(i = 0)$ or $(0 < i < m$ and $\lambda_i \leq \lambda_{i+1})$. Hence, in this case, γ is also produced by the operator $(-1)^{i+1}\overline{h}_{i+1}\overline{e}_{m-i-1}$. □

Remark 5.5. In the framework of combinatorial Hopf algebras in [1], the operator

$$\zeta := \sum_{m=0}^{\infty}\overline{h}_m$$

based on increasing chains is viewed as a multiplicative functional on the Hopf algebra $\mathcal{H}P$ associated to a labeled poset P. Indeed, we can define $\zeta([x,y])$ to be the coefficient of y in $x{\cdot}\zeta$; hence $\zeta([x,y])$ is the number of increasing saturated chains from x to y. Such a functional is called a *zeta function* for the corresponding Hopf algebra, and the pair $(\mathcal{H}P,\zeta)$ is called a combinatorial Hopf algebra. Similarly, $\zeta' := 1-H$ defined in (5.3) can be viewed as a different zeta function. According to Proposition 5.4, the linear functional

$$\mu := \sum_{m=0}^{\infty}(-1)^m\overline{e}_m$$

counting decreasing chains is the convolutional inverse of ζ; this means $\mu = \zeta \circ S$, where S is the antipode of $\mathcal{H}P$. The functional μ is called the *Möbius function* of $(\mathcal{H}P,\zeta)$.

The first family of Pieri operators we define is suggested by Theorem 3.1 (the notation is the same as in Section 3). Thus, we take our poset to be S_∞ with Bruhat order, fix a positive integer k, and set

$$v{\cdot}\overline{h}_m := \sum_{\substack{w\in P_k(v)\\ l(w)-l(v)=m}} \sigma_k(w,v)w\,. \tag{5.6}$$

These operators do not commute, as shown in the example below; hence the corresponding quasi-symmetric functions $K_{[v,w]}$ are not symmetric.

Example 5.7. For $n = 4$ and $k = 1$, we have

$$[1,3,2,4]{\cdot}\overline{h}_2 = -[3,2,1,4]\,, \quad \text{whence} \;\; [1,3,2,4]{\cdot}\overline{h}_2\overline{h}_1 = -[4,2,1,3]\,.$$

But

$$[1,3,2,4]\cdot\overline{h}_1 = [2,3,1,4] + [3,1,2,4]\,, \quad \text{whence } [1,3,2,4]\cdot\overline{h}_1\overline{h}_2 = 0\,.$$

The multiplication of a Grothendieck polynomial $\mathfrak{G}_v(x)$ by x_k is modeled by the operator $\sum_{i=1}^{\infty} \overline{h}_i$. But the quasi-symmetric functions $K_{[v,w]}$ encode more information than just the mentioned multiplication. Indeed, we have the following result.

Proposition 5.8. *The coefficient of $\mathfrak{G}_w(x)$ in the expansion of the product $x_k^p\,\mathfrak{G}_v(x)$ in the basis of Grothendieck polynomials is obtained by applying to $K_{[v,w]}$ the specialization map $f_p:\ \mathcal{Q}sym \to \mathbb{Z}$ defined by*

$$f_p(M_\alpha) := \begin{cases} 1 & \text{if } l(\alpha) = p \\ 0 & \text{otherwise}\,. \end{cases}$$

Proof. The argument is the same as the one used in the first proof of Proposition 5.4. □

Let us now define a family of Pieri operators suggested by Theorem 4.3 (the notation is the same as in Section 4). Thus, we fix a positive integer k, take our poset to be S_∞ with k-Bruhat order, and set

$$v\cdot\overline{h}_m := \sum_{\substack{w\in\Pi_k(v)\\ l(w)-l(v)=m}} w\,. \tag{5.9}$$

Based on Proposition 4.8, we can see that this operator is a descent Pieri operator; indeed, we label a cover $w < wt_{ab}$ (with $a<b$) of the k-Bruhat order with the pair (a,b), and use the order on pairs defined in (4.1).

Our descent Pieri operators (5.9) do not commute, whence the corresponding quasi-symmetric functions are not symmetric; indeed, Example 5.7 shows this too. On the other hand, the multiplication of a Grothendieck polynomial $\mathfrak{G}_v(x)$ by $\mathfrak{G}_{s_k}(x)$ is modeled by the operator H defined in (5.3). The quasi-symmetric functions $K_{[v,w]}$ encode more information than just the mentioned multiplication. For instance, it is easy to derive a similar result to Proposition 5.8 related to the multiplication of a Grothendieck polynomial $\mathfrak{G}_v(x)$ by $\mathfrak{G}_{s_k}(x)^p$.

Proposition 5.4 immediately implies an inverse formula to our Monk-type formula in Theorem 4.3; note that the former has no negative signs, unlike the latter. In order to state the inverse formula, let $\Delta_k(v)$ be the set of all decreasing saturated chains starting at v, including the chain of length 0; the total order on labels is still the one defined in (4.1). Let also $Q_k(v)$ consist of all permutations $\mathrm{end}(\gamma)$ for γ in $\Delta_k(v)$.

Theorem 5.10. *The following identity holds in the algebra of formal power series $\mathbb{Z}[[x_1,x_2,\ldots]]$, where $y_i := 1/(1-x_i)$:*

$$(y_1\ldots y_k)\,\mathfrak{G}_v(x) = \sum_{w\in Q_k(v)} \mathfrak{G}_w(x)\,.$$

Proof. According to (4.5), we have

$$\frac{1}{1-\mathfrak{G}_{s_k}(x)} = y_1\ldots y_k\,.$$

Hence the above multiplication is modeled by the operator $(1-H)^{-1}$, and, thus, Proposition 5.4 applies. We only have to prove that the formula obtained in this way is multiplicity free, and this is done in the next Proposition. $\square$

Proposition 5.11. *For any w in $Q_k(v)$, there is a unique chain γ in $\Delta_k(v)$ with* $\mathrm{end}(\gamma) = w$.

Proof. Let γ be a chain in $\Delta_k(v)$ with $\mathrm{end}(\gamma) = w$. As in the proof of Proposition 4.8, we represent γ as a sequence of transpositions $t := (t_{a_1 b_1}, \dots, t_{a_1 b_r}, t_{a_2 c_1}, \dots, t_{a_2 c_s}, \dots)$. We now have $a_1 > a_2 > \dots$, $b_1 < \dots < b_r$, $c_1 < \dots < c_s$ etc.

Positions $a_1, a_2, \dots$ are clearly determined by v and w only. So is position b_r; in fact $b_r = v^{-1}(w(a_1))$. We claim that

$$b_1 = \min \left\{ i \ : \ k < i \le b_r \quad \text{and} \quad w(a_1) < w(i) \le w(b_r) \right\}.$$

Similarly, for $1 \le p < r$, we claim that

$$b_{p+1} = \min \left\{ i \ : \ b_p < i \le b_r \quad \text{and} \quad w(b_p) < w(i) \le w(b_r) \right\}.$$

Indeed, denote the minimum by m, and assume that $b_{p+1} > m$ (we cannot have $b_{p+1} < m$, because b_{p+1} must clearly lie in the above set). Since $w(b_p) < w(m) < w(b_r)$ and $(w t_{a_1 b_1} \dots t_{a_1 b_p})(a_1) = w(b_p)$, it follows that one of the transpositions $t_{a_1 b_{p+1}}, \dots, t_{a_1 b_r}$ must increase length by more than 1 (cf. the necessary and sufficient condition for a transposition to increase length by 1, stated in the proof of Proposition 4.8). This is a contradiction, and therefore all positions b_i are uniquely determined by v and w. The uniqueness of $c_1, \dots, c_s, \dots$ follows by an inductive argument. $\square$

Remarks 5.12.

(1) The proof of Proposition 5.11 provides a similar algorithm to 4.9 for constructing the unique chain in $\Delta_k(v)$ from a permutation v to some permutation w in $Q_k(v)$.
(2) It is not hard to see that the lengths of increasing chains in k-Bruhat order starting at any given permutation in S_∞ are bounded, but the lengths of decreasing chains are not. This explains why the summation in Theorem 4.3 is finite, whereas the one in Theorem 5.10 is infinite.

Finally, we present a new formula for the restriction of a dominant line bundle to a Schubert variety. This is in terms of chains in the k-Bruhat order, and, unlike all the other formulas in this paper, it is not multiplicity free. As stated in the introduction, it is not clear how to relate this formula to the one in [9] in terms of Young tableaux, and the more general one in [21], based on Littelmann paths. The advantages of our formula are the fact that it is straightforward to invert, and the fact that it is an identity in an algebra of formal power series, not just for Schubert classes in K-theory.

Recall that $y_i := 1/(1-x_i)$ are representatives for the classes $[L_i]$ of the tautological line bundles V_i/V_{i-1} on the flag variety Fl_n (or, more generally, on the flag bundle $Fl(V)$, cf. Section 2). A dominant line bundle is one of the form $L^\lambda := L_1^{\otimes \lambda_1} \dots L_n^{\otimes \lambda_n}$, where $\lambda := (\lambda_1 \ge \dots \ge \lambda_n \ge 0)$ is a partition. Its class in K-theory is represented by the monomial $y^\lambda := y_1^{\lambda_1} \dots y_n^{\lambda_n}$. Let $\mu := (\mu_1 \ge \dots \ge \mu_k > 0)$ be the conjugate partition of λ.

Corollary 5.13. *The following identity holds in the algebra of formal power series* $\mathbb{Z}[[x_1, x_2, \ldots]]$, *where* $y_i := 1/(1 - x_i)$*:*

$$y^\lambda \, \mathfrak{G}_v(x) = \sum_\gamma \mathfrak{G}_{\mathrm{end}(\gamma)}(x) \,,$$

where the summation is over all chains of the form

$$\gamma \ : \ v = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \ \cdots \ \xrightarrow{\gamma_k} \ w_k = \mathrm{end}(\gamma) \,,$$

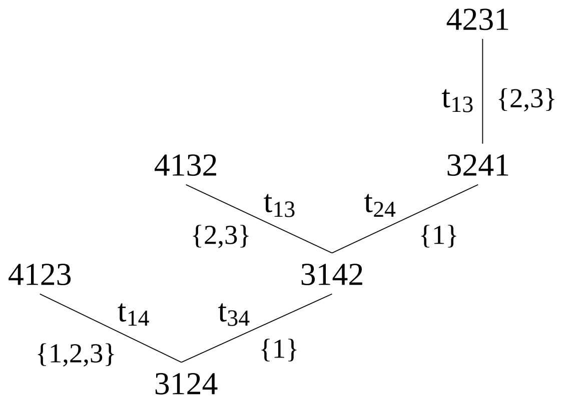
where γ_i *is a (decreasing saturated) chain in* $\Delta_{\mu_i}(w_{i-1})$ *with end*$(\gamma_i) = w_i$*, for* $i = 1, \ldots, k$.

Proof. Just notice that $y^\lambda = (y_1 \ldots y_{\mu_1}) \ldots (y_1 \ldots y_{\mu_k})$. We can now apply Theorem 5.10 repeatedly. □

Example 5.14. We reproduce the example on page 721 in [9], working with Grothendieck polynomials in only one set of variables instead. We have

$$\begin{aligned}(y_1^3 y_2 y_3) \, \mathfrak{G}_{3124}(x) = \mathfrak{G}_{3124}(x) + (3\mathfrak{G}_{4123}(x) + \mathfrak{G}_{3142}(x)) + \\ + (2\mathfrak{G}_{4132}(x) + \mathfrak{G}_{3241}(x)) + 2\mathfrak{G}_{4231}(x) + \ldots \,.\end{aligned}$$

The brackets contain the Grothendieck polynomials indexed by permutations of the same length, which have the same degrees of their lowest homogeneous components. In the right hand side of the formula, we only indicated the Grothendieck polynomials indexed by permutations in S_4; indeed, if we are working in $K^0(Fl_4)$, as in [9], the others do not appear. Here $\lambda = \mu = (3, 1, 1)$, so we have three subchains, in 3-, 1-, and 1-Bruhat orders, respectively. The following tree illustrates the way in which the permutations in the formula are constructed; the edges are labeled by the corresponding transpositions, as well as by the numbers of the subchains (1, 2, or 3) to which those transpositions can possibly belong.



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