# LAGRANGE INVERSION AND SCHUR FUNCTIONS

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Abstract. Macdonald defined an involution on symmetric functions by considering the Lagrange inverse of the generating function of the complete homogeneous symmetric functions. The main result we prove in this note is that the images of skew Schur functions under this involution are either Schur positive or Schur negative symmetric functions. The proof relies on the combinatorics of Lagrange inversion. We also present a $q$-analogue of this result, which is related to the $q$-Lagrange inversion formula of Andrews, Garsia, and Gessel, as well as the operator $\nabla$ of Bergeron and Garsia.

**Keywords**: Lagrange inversion, Schur functions, Dyck paths, Macdonald polynomials.

## 1. Introduction

In this note, we use the terminology and notation in [18]. In particular, $h_\lambda$ denotes the complete homogeneous symmetric function indexed by the partition $\lambda$, and $H(t) := \sum_{n=0}^{\infty} h_n t^n$.

Let us consider the following involution on the ring $\Lambda$ of symmetric functions:

$$h_\lambda \mapsto \psi(h_\lambda) = h_\lambda^* := h_{\lambda_1}^* h_{\lambda_2}^* \dots ,$$

where $h_n^*$ are defined by the condition that $tH^*(t) = t + h_1^* t^2 + h_2^* t^3 + \dots$ is the compositional inverse of $tH(-t)$. This is essentially the involution considered by Lascoux ([16] (6.3)) and Macdonald ([18] page 35); in fact, Macdonald's $h_n^*$ differs from the one defined above by a factor of $(-1)^n$. Note that the involution $\psi$ is related to composition of power series in the same way as the standard involution is related to multiplication of power series. On the other hand, the involution $\psi$ is closely related to the operator $\nabla$ on $\Lambda[q,t]$ defined in [2], which is discussed in Section 4. Let us also note that $h_\lambda^*$ form a basis of $\Lambda$. The importance of this basis was highlighted by its relation to the top connection coefficients in the center of the group algebra of the symmetric group (see [7], [18] p. 132, and [13]).

Let $E(t) := \sum_{n=0}^{\infty} e_n t^n$. It is well-known that $E(t)H(-t) = 1$. Recall the Lagrange inversion formula (see e.g. [12] or [22]), which asserts (in one of its equivalent forms) that the compositional inverse $tG(t)$ of a formal power series $tF(t)$ with $F(0) \neq 0$ satisfies

$$[t^n]\, G(t)^k = \frac{k}{n+k}\, [t^n]\, F(t)^{-n-k} \,. \tag{1.1}$$

Setting $F(t)=H(-t)=E(t)^{-1}$, $G(t)=H^*(t)$, and $k=1$, we obtain

$$h_n^* = \sum_{\lambda\vdash n}(-1)^{n-l(\lambda)}\frac{1}{n+1}\binom{n+l(\lambda)}{n,\, m_1(\lambda),\, m_2(\lambda),\ldots}h_\lambda \tag{1.2}$$
$$= \sum_{\lambda\vdash n}\frac{1}{n+1}\binom{n+1}{m_0(\lambda)+1,\, m_1(\lambda),\, m_2(\lambda),\ldots}e_\lambda\,;$$

here $m_k(\lambda)$ denotes the multiplicity of part $k$ in $\lambda$, and $m_0(\lambda)=n-\sum_{k\geq 1}m_k(\lambda)$. Setting $F(t)=E(t)^{-1}$, $G(t)=H^*(t)$, and $k=-1$, we obtain the following formula for $e_n^*:=\psi(e_n)$, which is also derived in [18]:

$$e_n^* = (-1)^{n-1}\sum_{\lambda\vdash n}\frac{1}{n-1}\binom{n-1}{m_0(\lambda)-1,\, m_1(\lambda),\, m_2(\lambda),\ldots}e_\lambda\,. \tag{1.3}$$

Note that Macdonald also obtains a formula for $p_n^*:=\psi(p_n)$. On the other hand, we can express $h_n^*$ in the basis of Schur functions by using the Cauchy formula (cf. [14] and [21]), and obtain a similar formula for $e_n^*$. Indeed, one form of the Cauchy formula may be written

$$\frac{1}{H(-y_1t)\ldots H(-y_mt)} = \sum_\lambda s_{\lambda'}(y_1,\ldots,y_m)s_\lambda t^{|\lambda|}\,,$$

where $\lambda'$ is the conjugate partition to $\lambda$. So taking $y_1=\ldots=y_m=1$, $m=n+k$, and then applying (1.1) with $k=1$ and $k=-1$, yields

$$h_n^* = \sum_{\lambda\vdash n}\frac{s_{\lambda'}(1^{n+1})}{n+1}s_\lambda\,, \qquad e_n^* = (-1)^{n-1}\sum_{\lambda\vdash n}\frac{s_{\lambda'}(1^{n-1})}{n-1}s_\lambda\,; \tag{1.4}$$

here $s_\lambda(1^k)$ denotes the number obtained by specializing the first $k$ variables in $s_\lambda$ to 1, and the rest of the variables to 0.

The symmetric functions $h_n^*$ are related to various combinatorial objects. For instance, formula (1.2) can be expressed combinatorially in terms of trees and *parking functions* (see below, and also [15, 22, 21]); the latter are sequences $(\alpha_1,\ldots,\alpha_n)$ containing at least $k$ entries less than or equal to $k$, for all $1\leq k\leq n$. The expansion of $h_n^*$ in terms of monomial symmetric functions also has several combinatorial interpretations (see [21] and [19]). As shown by Stanley in [21], $h_n^*$ is the so-called *flag symmetric function* of the *noncrossing partition lattice*. On the other hand, Haiman showed in [14] that $h_n^*$ is the Frobenius chracteristic of the representation of the symmetric group on the set of parking functions tensored with the sign representation. Stanley realized the same representation in [21] as a so-called *local action* of the symmetric group on the maximal chains of the noncrossing partition lattice.

In this note, we present combinatorial proofs of (1.3) and (1.4), some related facts (including a reference to certain analogs of parking functions which we define), and prove that the involution $\psi$ maps *any* skew Schur function to a Schur positive or a Schur negative symmetric function. Finally, we generalize the latter result by proving a special case of the main conjecture in [3] concerning the operator $\nabla$. The author is grateful to A. Garsia for introducing him to the operator $\nabla$, and to one of the referees for suggesting many improvements in Sections 1 and 2.

## 2. The Combinatorics of the Formulas for $h_n^*$ and $e_n^*$

We will consider sequences $(a_1, \ldots, a_{n+1})$ of nonnegative integers satisfying the property

$$\sum_{i=1}^{k} a_i \geq k\,, \quad k = 1, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n+1} a_i = n\,, \tag{2.1}$$

as well as sequences $(a_1, \ldots, a_{n-1})$ of nonnegative integers satisfying the property

$$\sum_{i=1}^{k} a_i > k\,, \quad k = 1, \ldots, n-1 \quad \text{and} \quad \sum_{i=1}^{n-1} a_i = n\,. \tag{2.2}$$

Note that in the first case we necessarily have $a_{n+1} = 0$. It is easy to see that the number of sequences of the first type is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, and that the number of sequences of the second type is $C_{n-1}$. Indeed, one can easily construct a bijection from sequences of the first type to Dyck paths from $(0,0)$ to $(2n,0)$ (every entry $a_i$, $1 \leq i \leq n$, corresponds to $a_i$ steps $(1,1)$ followed by one step $(1,-1)$), as well as a bijection from sequences of the first type with $n$ replaced by $n-1$ to sequences of the second type (just add 1 to the first entry and remove the last 0).

The following Lemma, often called the "cycle lemma", is due to Dvoretzky and Motzkin ([6], see also [5]), and was rediscovered many times. It can be used to prove various results, such as Lagrange inversion (cf. [20]), the formula for $C_n$, the fact that the number of parking functions of length $n$ is $(n+1)^{n-1}$ etc.; we will use it in the combinatorial proofs of (1.3) and (1.4).

**Lemma 2.3.**

1. *Among all $n+1$ distinct sequences obtained by cyclically permuting a sequence of $n+1$ nonnegative integers summing up to $n$, there is a unique one of the form (2.1).*
2. *Among all $n-1$ distinct sequences obtained by cyclically permuting a sequence of $n-1$ nonnegative integers summing up to $n$, there is a unique one of the form (2.2).*

It is useful to express $h_n^*$ combinatorially as follows:

$$h_n^* = \sum_{a} e_{\lambda(a)}\,, \tag{2.4}$$

where the summation ranges over all sequences $a = (a_1, \ldots, a_{n+1})$ of the form (2.1), and $\lambda(a)$ is the partition whose parts are the nonzero entries of $a$. This formula follows directly from (1.2) and Lemma 2.3 (1). Alternatively, we can find it in Raney's combinatorial proof of Lagrange inversion [20].

Before proceeding with the combinatorial proof of (1.3), which we have already derived from (1.1) by setting $k = -1$, let us note that Raney's combinatorial proof of (1.1) only works for $k > 0$.

*Combinatorial proof of (1.3).* Applying $\psi$ to the identity $H(t) = E(-t)^{-1}$, we obtain

$$\sum_{n=0}^{\infty} h_n^* t^n = \left( \sum_{n=0}^{\infty} (-1)^n e_n^* t^n \right)^{-1}\,. \tag{2.5}$$

Now define $e_n^{\#}$ by

$$e_n^{\#} = \sum_a e_{\lambda(a)}\,, \tag{2.6}$$

where the summation ranges over all sequences $a = (a_1, \ldots, a_{n-1})$ of the form (2.2). We claim that

$$\sum_{n=0}^{\infty} h_n^* t^n = \left(1 - \sum_{n=1}^{\infty} e_n^{\#} t^n\right)^{-1}. \tag{2.7}$$

Comparing (2.7) with (2.5) shows that $e_n^* = (-1)^{n-1} e_n^{\#}$ for $n \geq 1$, whence we have (1.3) by Lemma 2.3 (1). Formula (2.7) is equivalent to

$$h_n^* = \sum_\gamma e^{\#}_{\lambda(\gamma)}\,,$$

where the summation is over all compositions $\gamma$ of $n$. This formula has a simple combinatorial proof based on (2.4) and (2.6). Indeed, the right-hand side can be written as a summation over concatenations of sequences of the form (2.2), where the order of the concatenation is specified by $\gamma$. Now observe that every sequence of the form (2.1) with the final zero dropped can be decomposed uniquely as a concatenation of sequences of the form (2.2) with a final zero added. This gives the right bijection between the sequences indexing the two summations whose equality we want to prove. $\square$

*Combinatorial proof of (1.4).* We can also derive (1.4) from (2.4) in a combinatorial way. Expressing the right-hand side of (2.4) in the basis of Schur functions, we have that the coefficient of $s_\lambda$ is the number of semistandard Young tableaux of shape $\lambda'$ for which the type is a sequence of the form (2.1). Formula (1.4) now follows from Lemma 2.3 (1) and the combinatorial definition of Schur functions. The analogous formula for the expansion of $e_n^*$ follows in a similar way using Lemma 2.3 (2). $\square$

We conclude this section with some remarks concerning the representations of the symmetric group with Frobenius characteristic $h_n^*$ and $(-1)^{n-1}\, e_n^*$. Note that the parking functions of length $n$ are precisely the sequences of length $n$ containing $a_1$ 1's, ..., $a_n$ $n$'s, where $(a_1, \ldots, a_{n+1})$ satisfies property (2.1). We can define analogs of parking functions as sequences of length $n$ containing $a_1$ 1's, ..., $a_{n-1}$ $n-1$'s, where $(a_1, \ldots, a_{n-1})$ satisfies property (2.2). By Lemma 2.3 (2), there are $(n-1)^{n-1}$ such sequences. The symmetric group acts on them, and the Frobenius characteristic of the corresponding representation tensored with the sign representation is $(-1)^{n-1}\, e_n^*$ (the proof is similar to the one in [14] for $h_n^*$).

Finally, we note that the Robinson-Schensted correspondence establishes a bijection between parking functions of length $n$ and pairs $(S, R)$ of tableaux of the same shape $\lambda \vdash n$, with $R$ standard and $S$ semistandard, such that the type of $S$ is a sequence of the form (2.1). The same result is true for analogs of parking functions and sequences of the form (2.2). The above remarks justify combinatorially the identities

$$\sum_{\lambda \vdash n} \frac{s_\lambda(1^{n+1})}{n+1}\, f^\lambda = (n+1)^{n-1}\,, \qquad \sum_{\lambda \vdash n} \frac{s_\lambda(1^{n-1})}{n-1}\, f^\lambda = (n-1)^{n-1} \tag{2.8}$$

involving the dimensions of the representations corresponding to the symmetric functions in (1.4).

## 3. The Images of Skew Schur Functions under the Involution $\psi$

We prove the following result concerning the images of skew Schur functions under the involution $\psi$. Recall that the Jacobi-Trudi formula expresses the skew Schur function $s_{\lambda/\mu}$ as the determinant of the $l(\lambda)\times l(\lambda)$ matrix whose $(i,j)$-th entry is $h_{\lambda_i-i-\mu_j+j}$; here we adopt the convention that $h_m=0$ if $m<0$.

**Theorem 3.1.** *Given partitions $\mu\subseteq\lambda$, the symmetric function $(-1)^{i(\lambda/\mu)}\psi(s_{\lambda/\mu})$ is a nonnegative integer combination of Schur functions, where $i(\lambda/\mu)$ is the number of nonzero entries below the main diagonal in the Jacobi-Trudi matrix for $\lambda/\mu$. In particular, $i(\lambda)$ is the number of boxes below the diagonal in the Young diagram of $\lambda$.*

*Proof.* Let $n$ be the weight of $\lambda/\mu$. For the purposes of this proof, we use the French notation for partitions, that is $\lambda=(\lambda_1,\ldots,\lambda_k)$ with $\lambda_1\le\lambda_2\le\ldots\le\lambda_k$. With this notation, we have that the skew Schur function $s_{\lambda/\mu}$ is the determinant of the $k\times k$ matrix whose $(i,j)$-th entry is $h_{\lambda_i+i-\mu_j-j}$ (by the Jacobi-Trudi formula). The permutations with a nonzero contribution to this determinant are precisely those satisfying

$$\lambda_{\pi(i)}+\pi(i)\ge\mu_i+i\,,\quad 1\le i\le k\,. \tag{3.2}$$

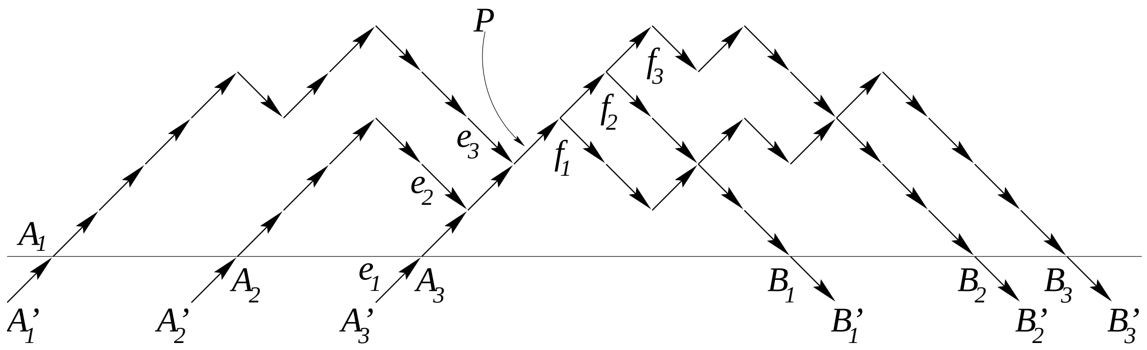
Consider $2k$ points on the $x$-axis in the plane, namely $A_i(2(\mu_i+i),0)$ and $B_i(2(\lambda_i+i),0)$, for $1\le i\le k$. For a given permutation $\pi$ satisfying (3.2), we will consider $k$-tuples of Dyck paths $P^\pi=(P_1^\pi,\ldots,P_k^\pi)$, where $P_i^\pi$ is a Dyck path from $A_i$ to $B_{\pi(i)}$ (in particular, a path reduces to a single point if $A_i=B_{\pi(i)}$). Combining the Jacobi-Trudi formula with (2.4) and the bijection from sequences satisfying (2.1) to Dyck paths, we obtain

$$\omega(\psi(s_{\lambda/\mu}))=\sum_\pi\varepsilon(\pi)\sum_{P^\pi}h_{\lambda(P^\pi)}\,; \tag{3.3}$$

here $\omega$ is the standard involution on symmetric functions, the first summation ranges over permutations satisfying (3.2), $\varepsilon(\pi)$ is the sign of $\pi$, and $\lambda(P^\pi)$ is the partition of $n$ whose parts are the lengths of the northeast steps in the paths $P_1^\pi,\ldots,P_k^\pi$.

We now define the number $i(P^\pi)$ of crossings between pairs of paths in $P^\pi$. The easiest way to do this is to introduce auxilliary points $A_i'(2(\mu_i+i)-1,-1)$ and $B_i'(2(\lambda_i+i)+1,-1)$. We add to every path $P_i^\pi$ the edges $A_i'A_i$ and $B_{\pi(i)}B_{\pi(i)}'$; then, for every pair of (augmented) paths we contract all the edges they have in common, and count all points of intersection which are not endpoints and in which neither path changes direction. By switching the paths at each crossing, we obtain a new configuration of paths $P^{\pi_0}$, corresponding to a permutation $\pi_0$. Clearly, $\varepsilon(\pi_0)=\varepsilon(\pi)(-1)^{i(P^\pi)}$. On the other hand, since there are no crossings of paths in $P^{\pi_0}$, we have that $\pi_0(i)<\pi_0(j)$ if and only if $\lambda_i+i<\mu_j+j$, for every $i<j$. This condition characterizes $\pi_0$ and shows that it has $i(\lambda/\mu)$ inversions, whence $\varepsilon(\pi_0)=(-1)^{i(\lambda/\mu)}$ (since we use the French notation for partitions, we need to consider the number of nonzero entries *above* the main diagonal in the Jacobi-Trudi matrix for $\lambda/\mu$). As an aside, we note that the path switching argument above also shows that $\pi\le\pi_0$ in Bruhat order. In fact, more is true, namely that the

permutations satisfying (3.2) are precisely those in the interval $[\widehat{0}, \pi_0]$, where $\widehat{0}$ is the identity permutation (see for instance [23]).




We now change the order of summation in (3.3). Let us consider the set $\mathcal{G}$ of all plane directed graphs $G$ for which there is a permutation $\pi$ and a $k$-tuple of paths $P^\pi$ as above whose union is $G$ (the edges of the paths are now assumed to be directed, and they correspond to steps $(1,1)$ or $(1,-1)$); in fact, it is enough to consider only $\pi = \pi_0$ in this definition. According to the remarks above, we have

$$(-1)^{i(\lambda/\mu)}\,\omega(\psi(s_{\lambda/\mu})) = \sum_{G \in \mathcal{G}} \sum_{\substack{\pi, P^\pi : \\ \bigcup_i P_i^\pi = G}} (-1)^{i(P^\pi)}\, h_{\lambda(P^\pi)}\,. \tag{3.4}$$

Now let $G$ be a fixed graph in $\mathcal{G}$, and let $\overline{G}$ be the graph obtained from it by adding the auxilliary directed edges $A'_iA_i$ and $B_iB'_i$. Let us weight the edges of $\overline{G}$ by a weight function $w$ such that all auxilliary edges have weight 1 and the weight satisfies the property of flow conservation at every vertex with nonnegative $y$-coordinate. Consider an arbitrary maximal path $P$ in $G$ consisting only of edges oriented northeast. Let $E(P) := (e_1, \ldots, e_m)$ be the *sequence* of edges in $\overline{G}$ with heads belonging to $P$; these edges are ordered by the $y$-coordinate of their tails, and every edge is repeated $w(e)$ times. Similarly, we consider the *multiset* $F(P) := \{f_1, \ldots, f_m\}$ of edges in $\overline{G}$ with tails belonging to $P$ (see the figure above). Let $x(e_i)$ (respectively $x(f_i)$) denote the $x$-coordinate of the head of $e_i$ (respectively tail of $f_i$). In order to find all possible $P^\pi$ with $\bigcup_i P_i^\pi = G$ for which every edge $e$ appears in exactly $w(e)$ paths, it is enough to consider for every path $P$ specified above all permutations $(f'_1, \ldots, f'_m)$ of the multiset $F(P)$ such that $x(e_i) \le x(f'_i)$ for $1 \le i \le m$; indeed, we form paths starting with $e_i$ followed by a certain subpath of $P$ and $f'_i$, then we join these paths in the obvious way. We note the following facts.

1. The above procedure is carried out independently for every $P$, so we need to take the product of the contributions of all $P$ to the second sum in the right-hand side of (3.4).
2. Assuming that $F(P)$ is a set rather than a multiset, the contribution of a given $P$ is 0 unless all integers $x(e_i)$ are different; in the latter case, the contribution is the skew Schur function $s_{\nu/\rho}$, where $\rho_i = x(e_i) - i$ and $\nu_i = x(f_i) - i$ for $1 \le i \le m$ (by the Jacobi-Trudi formula).

3. If at least one edge of $G$ oriented southeast has weight greater than 1 (if this happens for some $w$, it happens for all $w$), then we can find a path $P$ with no repeated edges in $F(P)$ and repeated edges in $E(P)$; hence the contribution of such $G$ to the right-hand side of (3.4) is 0.

Hence, we can restrict the summation in the right-hand side of (3.4) to the set of graphs $\mathcal{G}_0$ for which a vertex of indegree 2 and outdegree 1 necessarily has the edge starting at it oriented northeast, and is not among the $A_i$'s. Clearly, there is a unique way to weight these graphs, and the weight of every edge oriented southeast is 1. By the above remarks, $(-1)^{i(\lambda/\mu)}\,\omega(\psi(s_{\lambda/\mu}))$ can be written as a sum over $\mathcal{G}_0$ of products of skew Schur functions, which is obviously a nonnegative integer combination of Schur functions.

□

We conclude this section by pointing out the interesting open problem of finding nice combinatorial interpretations for the coefficients of the expansion of $\psi(s_\lambda)$ in the basis of Schur functions. In other words, we are asking for generalizations of the formulas (1.4).

## 4. The Operator $\nabla$ of Bergeron and Garsia

In this section we present a generalization of Theorem 3.1 in terms of the operator $\nabla$ defined by F. Bergeron and A. Garsia (see [3, 2]). This operator acts on $\Lambda[q,t]$, and has a modified version of the Macdonald polynomials as eigenfunctions. More precisely, for every partition $\mu$ of $n$, we define

$$\widetilde{H}_\mu := \sum_{\lambda\vdash n} s_\lambda\, \widetilde{K}_{\lambda\mu}(q,t)\,,$$

where

$$\widetilde{K}_{\lambda\mu}(q,t) := t^{n(\mu)}\, K_{\lambda\mu}(q,1/t)\,,$$

$K_{\lambda\mu}(q,t)$ are the Macdonald $q,t$-Kostka coefficients defined in [17], and $n(\mu) := \sum_i (i-1)\mu_i$. Macdonald conjectured in [17] that $K_{\lambda\mu}(q,t)$ are polynomials in $q,t$ with positive integer coefficients. Recently it was shown that they are polynomials with integer coefficients, but the positivity still remains to be proved. With this notation, the operator $\nabla$ is defined by

$$\nabla \widetilde{H}_\mu := T_\mu \widetilde{H}_\mu\,,$$

where

$$T_\mu := t^{n(\mu)} q^{n(\mu')}\,.$$

The following is one of the main conjectures [8, 3] concerning the operator $\nabla$; it is based on thorough computer experiments.

**Conjecture 4.1.** *For every partition $\mu$ of $n$, we have the expansion*

$$\nabla\, s_\lambda = \varepsilon_\lambda \sum_{\mu\vdash n} s_\mu\, C_{\lambda\mu}(q,t)\,,$$

*where $\varepsilon_\lambda$ is a sign, and $C_{\lambda\mu}(q,t)$ are polynomials in $q,t$ with nonnegative integer coefficients.*

The fact that $C_{\lambda\mu}(q,t)$ are polynomials in $q,t$ with integer coefficients has been proved recently by A. Garsia by extending the machinery in [9], and will appear in [3]. Some of the coefficients $C_{\lambda\mu}(q,t)$ were identified. For instance, one obtains the $q,t$-Catalan sequence studied in [9] by setting $\lambda=\mu=(1^n)$. Other identities concerning the images of various symmetric functions under $\nabla$ are known. For example, according to Theorem 3.4 in [9], we have that

$$\nabla\, e_n = \mathrm{DH}_n(x;q,t)\,; \tag{4.2}$$

here $\mathrm{DH}_n(x;q,t)$ is the conjectured bigraded Frobenius characteristic of diagonal harmonics, which is given by formula (15) in [9], and is related to a bivariate version of Lagrange inversion. Since $\mathrm{DH}_n(x;1,1)=h_n^*$ (see [14] or the discussion below), we have that $\nabla_{q=t=1}=\psi\circ\omega$. This means that if Conjecture 4.1 is true, then $\varepsilon_\lambda=(-1)^{i(\lambda')}$, by Theorem 3.1. The next step towards Conjecture 4.1 is to set only $t=1$. This is an interesting special case, because the operator $\nabla_{t=1}$ is known to be multiplicative by formula (92) in [9], and hence one can combine the Jacobi-Trudi formula with (4.2) to compute $\nabla_{t=1}\, s_\lambda$. We also need to recall from [9] (cf. (41), (43), (55), and (90)) the fact that $\mathrm{DH}_n(x;q,1)$, which will be denoted by $h_n^*(q)$, is given by the $q$-Lagrange inversion formula due to Andrews, Garsia, and Gessel [1, 10, 11]. More precisely, we have the following $q$-analogue of formula (2.4):

$$h_n^*(q)=\sum_a q^{\sum_i (n-i)a_i-\binom{n}{2}}\, e_{\lambda(a)}\,, \tag{4.3}$$

where the summation again ranges over all sequences $a=(a_1,\ldots,a_{n+1})$ of the form (2.1); the exponent of $q$ in the right-hand side of (4.3) is precisely half the area between the Dyck path with alternating steps and the Dyck path corresponding to the sequence $a$ under the bijection discussed at the beginning of Section 2. This given, M. Bousquet-Mélou, F. Bergeron, and D. Gouyou-Beauchamps [4] determined the coefficient $C_{\lambda,(1^n)}(q,1)$, and thus found the sign $\varepsilon_\lambda$ too. On the other hand, it turns out that our proof of Theorem 3.1 translates easily into a proof of the special case of Conjecture 4.1 corresponding to $t=1$.

**Theorem 4.4.** *Given partitions $\mu\subseteq\lambda$, the expansion of the symmetric function $(-1)^{i(\lambda/\mu)}\,\nabla_{t=1}\, s_{\lambda/\mu}$ in the basis of Schur functions involves only polynomials in $q$ with nonnegative integer coefficients.*

*Proof.* Define the multiplicative operator $\widetilde{\nabla}$ on $\Lambda[q]$ by $\widetilde{\nabla}\, h_n := h_n^*(q)$. By (4.2) and the multiplicativity of $\nabla_{t=1}$, we have $\nabla_{t=1}=\widetilde{\nabla}\circ\omega$. Hence it suffices to prove the Theorem for $\widetilde{\nabla}$ instead of $\nabla_{t=1}$.

The proof now proceeds in the same way as the proof of Theorem 3.1 until formula (3.4). We now make the crucial observation that the sum of areas between the Dyck paths in $P^\pi$ and the corresponding Dyck paths with alternating steps is invariant under switching paths at crossings. Hence the power of $q$ attached to the terms in the right-hand side of the $q$-analogue of (3.4) can be factored out of the second summation. The remaining part of the proof is identical. □

Theorem 4.4 suggests that Conjecture 4.1 might actually hold for $\nabla$ applied to skew Schur functions.

## References


[1] G. E. Andrews. Identities in combinatorics III: A $q$-analogue of the Lagrange inversion theorem. *Proc. Amer. Math. Soc.*, 53:240–245, 1975.

[2] F. Bergeron, N. Bergeron, A. Garsia, M. Haiman, and G. Tesler. Lattice diagram polynomials and extended Pieri rules. to appear in Adv. Math.

[3] F. Bergeron and A. Garsia. Identities and conjectures for a remarkable operator on symmetric polynomials. to appear in Séminaire Lotharingien de Combinatoire, Publ. Inst. Rech. Math. Av., Univ. Louis Pasteur, Strasbourg.

[4] M. Bousquet-Mélou, F. Bergeron, and D. Gouyou-Beauchamps. Personal communication, May 1998.

[5] N. Dershowitz and S. Zaks. The cycle lemma and some applications. *European J. Combin.*, 11:35–40, 1990.

[6] A. Dvoretzky and Th. Motzkin. A problem of arrangements. *Duke Math. J.*, 14:305–313, 1947.

[7] H. K. Farahat and G. Higman. The centres of symmetric group rings. *Proc. Royal Soc. (A)*, 250:212–221, 1959.

[8] A. Garsia. Personal communication, May 1998.

[9] A. Garsia and M. Haiman. A remarkable $q, t$-Catalan sequence and $q$-Lagrange inversion. *J. Algebraic Combin.*, 5:191–244, 1996.

[10] A. M. Garsia. A $q$-analogue of the Lagrange inversion formula. *Houston J. Math.*, 7:205–237, 1981.

[11] I. Gessel. A noncommutative generalization and $q$-analogue of the Lagrange inversion formula. *Trans. Amer. Math. Soc.*, 257:455–482, 1980.

[12] I. P. Goulden and D. M. Jackson. *Combinatorial Enumeration.* Wiley Intersci. Ser. in Discrete Math. John Wiley & Sons, 1983.

[13] I. P. Goulden and D. M. Jackson. Symmetric functions and Macdonald's result for top connection coefficients in the symmetric group. *J. Algebra*, 166:364–378, 1994.

[14] M. Haiman. Conjectures on the quotient ring by diagonal invariants. *J. Algebraic Combin.*, 3:17–76, 1994.

[15] M. Haiman and W. Schmitt. Incidence algebra antipodes and Lagrange inversion in one and several variables. *J. Combin. Theory Ser. A*, 50:172–185, 1989.

[16] A. Lascoux and M.-P. Schützenberger. Formulaire raisonné de fonctions symétriques. Publ. Univ. Paris 7, 1985.

[17] I. G. Macdonald. A new class of symmetric functions. volume 372 of *Séminaire Lotharingien de Combinatoire, Publ. Inst. Rech. Math. Av.*, pages 131–171. Univ. Louis Pasteur, Strasbourg, 1988.

[18] I. G. Macdonald. *Symmetric Functions and Hall Polynomials.* Oxford Mathematical Monographs. Oxford University Press, Oxford, second edition, 1995.

[19] I. Pak and A. Postnikov. Enumeration of trees and one amazing representation of the symmetric group. In D. Stanton, editor, *Eighth International Conference on Formal Power Series and Algebraic Combinatorics*, pages 385–389, University of Minnesota, Minneapolis, 1996.

[20] G. N. Raney. Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, 94:441–451, 1960.

[21] R. Stanley. Parking functions and noncrossing partitions. *Electronic J. Combin.*, 4 (Wilf Festschrift)(2):R20, 1997.

[22] R. P. Stanley. *Enumerative Combinatorics*, volume II. to appear.

[23] R. Winkel. On the expansion of Schur and Schubert polynomials into standard elementary monomials. to appear in *Adv. Math.*



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