COMBINATORIAL ASPECTS OF THE K-THEORY OF GRASSMANNIANS

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Abstract. In this paper we study Grothendieck polynomials indexed by Grassmannian permutations, which are representatives for the classes corresponding to the structure sheaves of Schubert varieties in the K-theory of Grassmannians. These Grothendieck polynomials are nonhomogeneous symmetric polynomials whose lowest homogeneous component is a Schur polynomial. Our treatment, which is closely related to the theory of Schur functions, gives new information about these polynomials. Our main results are concerned with the transition matrices between Grothendieck polynomials indexed by Grassmannian permutations and Schur polynomials on the one hand, and with Pieri formulas for these Grothendieck polynomials on the other hand.

1. Introduction

The singular cohomology of flag varieties has attracted a great interest among algebraic combinatorialists in recent years. The seminal work of Bernstein, Gelfand and Gelfand [1] on the cohomology of the spaces G/P, and Lascoux and Schützenberger [12], who defined the *Schubert polynomials* as representatives for Schubert classes in type A, was the starting point of a vast literature on Schubert calculus. The K-theory of flag varieties, which is the next level of complexity after singular cohomology in encoding the topological structure of these varieties, leads to a generalization of Schubert calculus. The analogs of Schubert polynomials, as representatives for the K-theory classes determined by the structure sheaves of Schubert varieties, are the *Grothendieck polynomials*. They were introduced by Lascoux and Schützenberger in [14], and studied more extensively in [11]. These polynomials are defined using the so-called *isobaric divided difference operators*, as opposed to the divided difference operators defining Schubert polynomials. The literature on the K-theory of flag varieties is modest compared with that on Schubert calculus. The main references are: [10, 3] (where the general structure of the K-theory of G/P is studied), [13, 11] (where the Grothendieck polynomials are defined and their main properties presented), [6] (where a combinatorial formula for the Grothendieck polynomials similar to the one in [2] is presented), and [8] (where a Pieri formula for Grothendieck polynomials is given).

In this paper we study Grothendieck polynomials corresponding to Grassmannian permutations, which represent Schubert classes in the K-theory of Grassmannians. They are nonhomogeneous symmetric polynomials whose lowest homogeneous component is a Schur polynomial. Our treatment, which is closely related to the theory of Schur functions, gives new information about these Grothendieck polynomials. Our main results are concerned with the transition matrices between Grothendieck polynomials indexed by Grassmannian permutations and Schur polynomials on the one hand, and with Pieri formulas for these Grothendieck polynomials on the other hand (note that the Pieri formula in [8] is not a polynomial identity, being valid only up to a certain ideal, and uses a different set of variables than the one below).

I am grateful to Alain Lascoux for useful conversations and several suggestions he made on a first version of this work.

2. Grothendieck Polynomials for Grassmannian Permutations and Schur Polynomials

Let $Gr_k(\mathbb{C}^n)$ denote the Grassmannian of k-dimensional subspaces in $\mathbb{C}^n$, which is the simplest complex flag variety (n and k are fixed throughout this paper). It is well-known that the singular cohomology of $Gr_k(\mathbb{C}^n)$ is a certain quotient of the ring of symmetric polynomials in $x_1, \ldots , x_k$, which is just the polynomial ring in the elementary symmetric polynomials $e_i = e_i(x_1, \ldots , x_k)$; the complete homogeneous symmetric polynomials $h_i = h_i(x_1, \ldots , x_k)$ are also polynomial generators. More precisely, $H^*(Gr_k(\mathbb{C}^n))$ is the quotient of $\mathbb{Z}[e_1, \ldots , e_k]$ (the elements e_i are now viewed as symbols) by the ideal generated by all $p \times p$ determinants $\det(e_{1+j-i})_{1\leq i,j\leq p}$, where $n - k + 1 \leq p \leq n$. Alternatively, the cohomology ring can be described as the quotient of the polynomial ring $\mathbb{Z}[e_1, \ldots , e_k, h_1, \ldots , h_{n-k}]$ by the ideal generated by relations which can be expressed in the following way:

$$\left(\sum_{i=0}^{k} e_i t^i\right)\left(\sum_{j=0}^{n-k}(-1)^j h_j t^j\right) = 1\,.$$

Recall that $Gr_k(\mathbb{C}^n)$ has a decomposition into Schubert cells, whose closures are the *Schubert varieties*. They are indexed by *Grassmannian permutations* of $\{1, \ldots , n\}$ which have their unique descent in position k, that is, permutations $w = (w_1, \ldots , w_n)$ with $w_i < w_{i+1}$ if and only if $i \neq k$ (throughout this paper we use the one-line notation for permutations). A more natural indexing set is formed by partitions of n (identified with their Young diagrams) with at most k rows and $n-k$ columns. All partitions considered in this paper have at most k nonzero parts, and we adopt the convention of appending extra parts equal to 0 in order to have exactly k parts. The correspondence between the Grassmannian permutations and the partitions mentioned above is given by

$$w \mapsto \lambda = (c_k(w), \ldots , c_1(w))\,,$$

where $c(w) = (c_1(w), \ldots , c_{n-1}(w))$ denotes the *code* of a permutation w; recall that $c_i(w)$ is defined to be the cardinality of the set $\{j > i \,:\, w_j < w_i\}$, and that $c_i(w) = w_i - i$ for $i \leq k$, if w is a Grassmannian permutation with descent in position k. The Grassmannian permutation corresponding to the partition λ will be denoted by $w(\lambda)$. There are cohomology classes (known as *Schubert classes*) corresponding to Schubert varieties, and, in fact, they form an integral basis for cohomology. The natural representatives for Schubert classes are the *Schur polynomials* $s_\lambda = s_\lambda(x_1, \ldots , x_k)$ indexed by the partitions mentioned above; they are special cases of *Schubert polynomials* (see e.g. [17] or [7]).

Now let us consider the K-theory of $Gr_k(\mathbb{C}^n)$, which is the Grothendieck ring of complex vector bundles over this Grassmannian. Using the λ-ring notation, we can describe $K^0(Gr_k(\mathbb{C}^n))$ as the quotient of the ring of Laurent polynomials in $a_1, \ldots , a_n$, which are symmetric in both $\{a_1, \ldots, a_k\}$ and $\{a_{k+1}, \ldots, a_n\}$, by the ideal generated by the relations

$$\Lambda^{n-k}(a_{k+1} + \ldots + a_n) = \ldots = \Lambda^n(a_{k+1} + \ldots + a_n) = 0\,;$$

see for instance [11]. There are K-theory classes which correspond to the structure sheaves of Schubert varieties; they are indexed by the same partitions as Schubert classes in singular cohomology, and they form a basis of the Grothendieck ring. The natural representatives for Schubert classes in K-theory are certain symmetric polynomials $\mathfrak{G}_\lambda(a_1, \ldots, a_k)$ introduced by Lascoux and Schützenberger, who called them Grothendieck polynomials; they are indexed by the partitions mentioned above. These polynomials are special cases of the more general Grothendieck polynomials indexed by permutations of $\{1, \ldots, n\}$, which are representatives for Schubert classes in the K-theory of the variety of complete flags in $\mathbb{C}^n$; in other words, we have

$$\mathfrak{G}_\lambda(a_1, \ldots, a_k) = \mathfrak{G}_{w(\lambda)}(a_1, \ldots, a_n)\,.$$

There is a general recursive procedure for computing Grothendieck polynomials in the variables $a_1, \ldots, a_n$, which is based on the isobaric divided difference operators π_i defined by

$$\pi_i\, f(a_1, \ldots, a_n) = \frac{a_i\, f(a_1, \ldots, a_n) - a_{i+1}\, f(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n)}{a_i - a_{i+1}}\,.$$

Given an arbitrary permutation w, we can associate with it an operator π_w by considering an arbitrary reduced word $(i_1, \ldots, i_{l(w)})$ for w (that is $s_{i_1} \ldots s_{i_{l(w)}} = w$, where s_i is the transposition $(i, i+1)$ and $l(w)$ denotes the length of w), and defining $\pi_w := \pi_{i_1} \ldots \pi_{l(w)}$; this definition does not depend on the choice of the reduced word. We can now define Grothendieck polynomials by

$$\mathfrak{G}_{w_0}(a_1, \ldots, a_n) = \prod_{i=1}^{n-1} (1 - 1/a_i)^{n-i}\,, \tag{2.1}$$

$$\mathfrak{G}_w(a_1, \ldots, a_n) = \pi_{w^{-1}w_0}\, \mathfrak{G}_{w_0}(a_1, \ldots, a_n)\,, \tag{2.2}$$

where $w_0 = (n, n-1, \ldots, 1)$ is the longest permutation in the symmetric group Σ_n.

It is easy to check that the operators π_i provide a faithful representation of the degenerate Hecke algebra $H_n(0)$. This is generated by $u_1, \ldots, u_{n-1}$, subject to the following relations:

$$\begin{aligned} &u_i^2 = -u_i\,, \\ &u_i u_j = u_j u_i \quad \text{if } |i-j| \geq 2\,, \\ &u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}\,, \end{aligned} \tag{2.3}$$

and has a basis consisting of elements which can be identified with permutations in Σ_n.

We now make the substitution $x_i = 1 - 1/a_i$ for $1 \leq i \leq k$, and define a ring homomorphism from the Grothendieck ring to the singular cohomology of $Gr_k(\mathbb{C}^n)$ by sending the elementary symmetric polynomials $e_i(x_1, \ldots, x_k)$ in $K^0(Gr_k(\mathbb{C}^n))$ to the corresponding elements e_i in the singular cohomology ring. It turns out that this map is an isomorphism. In geometric terms, it corresponds to identifying the Chern classes in K-theory and singular cohomology of the dual of the tautological vector bundle on $Gr_k(\mathbb{C}^n)$. From now on, we will view the Grothendieck polynomials representing Schubert classes in $K^0(Gr_k(\mathbb{C}^n))$

as polynomials in $x_1, \ldots, x_k$, and simplify their notation to $\mathfrak{G}_\lambda$; more generally, we will view any Grothendieck polynomial indexed by a permutation w in Σ_n as a polynomial in $x_1, \ldots, x_n$, and will denote it by $\mathfrak{G}_w$ (see [14, 13, 6]). The isobaric divided difference operator discussed above can be expressed in terms of the x_i's in the following way:

$$f(x_1, \ldots, x_k) \mapsto \frac{(1-x_{i+1})\, f(x_1, \ldots, x_k) - (1-x_i)\, f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k)}{x_i - x_{i+1}};$$

from now on we write π_i for *this* operator, and define π_w as usual, for every permutation w. We will also need the divided difference operators ∂_w, which are used to define Schubert polynomials (see [17] or [7]); recall that

$$\partial_i\, f(x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}},$$

and that ∂_w is defined in a similar way to π_w.

It follows from their definition that the Grothendieck polynomials are nonhomogeneous polynomials in the x_i's, and that their lowest homogeneous component is the corresponding Schubert polynomial. Hence the lowest homogeneous component of $\mathfrak{G}_w$ has degree $l(w)$. In particular, the lowest homogeneous component of $\mathfrak{G}_\lambda$ is the Schur polynomial s_λ, which has degree $|\lambda|$ (i.e., the weight of λ, defined as the sum of its parts). A general property of Grothendieck polynomials ([11] (2.2)) tells us that $\mathfrak{G}_\lambda$ is symmetric in $x_1, \ldots, x_k$. Another fact about Grothendieck polynomials, which was proved in [6], is that the coefficients of monomials in different homogeneous components alternate in sign. In some of the proofs below, it is more convenient to use the slightly modified version of $\mathfrak{G}_\lambda$ in which all monomials have positive coefficients; thus, we define

$$\widetilde{\mathfrak{G}}_\lambda := (-1)^{|\lambda|}\, \mathfrak{G}_\lambda(-x_1, \ldots, -x_k).$$

The following result shows that Grothendieck polynomials corresponding to Grassmannian permutations can be obtained from Schur polynomials by a simple λ-ring substitution. Before stating the result, we recall the definition of a flag Schur function indexed by an arbitrary composition α with k parts (see e.g. [17]):

$$s_\alpha(X_1 - Y_1, \ldots, X_k - Y_k) := \det(h_{\alpha_i - i + j}(X_i - Y_i))_{1 \le i,j \le k};$$

here X_i, Y_i are alphabets, $h_m(X-Y) = \sum_{i=0}^m h_i(X) h_{m-i}(-Y)$, $h_i(-Y) = (-1)^i e_i(Y)$, and $h_i(X), e_i(Y)$ denote the corresponding complete homogeneous and elementary symmetric functions in the variables of X and Y. Recall that in λ-ring notation an alphabet is identified with the formal sum of its elements, so it makes sense to write m for the alphabet with m elements, all of which are equal to 1. For instance, we have $e_i(m) = \binom{m}{i}$ and $h_i(m) = \binom{m+i-1}{i}$. In this sense, we write $h_i(X-m)$ for $h_i(X-Y)$ with Y being the special alphabet m. We also need to consider the staircase shapes

$$\delta := (k-1, k-2, \ldots, 1, 0) \quad \text{and} \quad \overline{\delta} := (0, 1, \ldots, k-1).$$

Theorem 2.4. *We have that*

$$\mathfrak{G}_\lambda = (-1)^{k(k-1)/2}\, s_{\lambda + \overline{\delta}}(X, X-1, \ldots, X-(k-1)),$$

where $X := x_1 + x_2 + \ldots + x_k$.

Proof. Let u be the permutation $(k, k-1, \ldots, 1, k+1, k+2, \ldots, n)$, and note that $w(\lambda)u$ is a *dominant* permutation, i.e., its code is a partition. Given this fact, we know from [11] Lemma 2.10 that

$$\mathfrak{G}_{w(\lambda)u} = x^{\lambda+\delta}, \tag{2.5}$$

where $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ for any composition α. From (2.2) and the fact that $l(w(\lambda)u) = l(w(\lambda)) + l(u)$, it follows that

$$\mathfrak{G}_\lambda = \pi_u \, \mathfrak{G}_{w(\lambda)u} \, . \tag{2.6}$$

Now, according to formula (1.8) in [14], we have

$$\pi_u \, f(x_1, \ldots, x_k) = \frac{1}{\Delta} \, A\left(f(x_1, \ldots, x_k) \prod_{i=1}^{k} (1 - x_i)^{i-1} \right) ; \tag{2.7}$$

here Δ is the Vandermonde, and A is the antisymmetrization operator

$$A(g(x_1, \ldots, x_k)) := \sum_{w \in \Sigma_k} (-1)^{l(w)} \, g(x_{w(1)}, \ldots, x_{w(k)}) \, .$$

Hence, up to a sign of $(-1)^{k(k-1)/2}$, the polynomial $\mathfrak{G}_\lambda$ can be obtained by applying the usual maximal divided difference operator ∂_u to $x^{\lambda+\delta} \prod_{i=1}^k (x_i - 1)^{i-1}$; indeed, it is well-known (see e.g. [7]) that ∂_u has a similar expression to (2.7), namely

$$\partial_u \, f(x_1, \ldots, x_k) = \frac{1}{\Delta} \, A(f(x_1, \ldots, x_k)) \, .$$

We claim that the polynomial $x^{\lambda+\delta} \prod_{i=1}^k (x_i - 1)^{i-1}$ can be expressed as the flag Schur polynomial

$$s_{\lambda+(k-1)^k}(X_1, X_2 - 1, \ldots, X_k - (k-1)) \, ,$$

where $(k-1)^k$ denotes the partition with k parts equal to $k-1$, and $X_i := x_1 + x_2 + \ldots + x_i$. The proof of this fact is based on the following identity:

$$h_i(X - Y) - x h_{i-1}(X - Y) = h_i((X - x) - Y) \, ,$$

where x is an element of X. Indeed, let us consider the determinant in the complete homogeneous symmetric polynomials expressing the above flag Schur polynomial. We subtract column i multiplied by x_1 from column $i+1$ for $i = 1, \ldots, k-1$, then we subtract column i multiplied by x_2 from column $i+1$ for $i = 2, \ldots, k-1$, and so on. The obtained matrix is lower triangular, and its i-th diagonal entry is

$$h_{\lambda_i+k-1}(x_i - (i-1)) = x_i^{\lambda_i+k-i} (x_i - 1)^{i-1} \, ,$$

whence the above claim. The desired expression for $\mathfrak{G}_\lambda$ now follows by applying Proposition 3.11 in [17]; indeed, according to this result we have

$$\partial_u \, s_\alpha(X_1 + Z_1, \ldots, X_k + Z_k) = s_{\alpha-\delta}(X_k + Z_1, \ldots, X_k + Z_k) \, ,$$

where Z_i are arbitrary alphabets not containing x_j, and X_i are as above. □

The following theorem gives an alternative expression for a Grothendieck polynomial corresponding to a Grassmannian permutation, namely as a linear combination of Schur polynomials. This expression is more useful in practical applications (such as our Pieri formulas in the next section) than the one in Theorem 2.4, because Schur polynomials are well understood. In order to state our result, we define $\widehat{\lambda}$ as the unique maximal partition with k rows (in the Young lattice) obtained from λ by adding at most $i-1$ boxes to its i-th row, for $2 \le i \le k$.

Theorem 2.8. *We have*

$$\mathfrak{G}_\lambda = \sum_{\lambda \subseteq \mu \subseteq \widehat{\lambda}} (-1)^{|\mu|-|\lambda|} g_{\lambda\mu}\, s_\mu \,.$$

Furthermore, every coefficient $g_{\lambda\mu}$ is nonnegative, and, in fact, is the number of row and column strict tableaux of shape μ/λ with entries in the i-th row (counted from the top row of λ and μ) restricted to $1, 2, \ldots, i-1$ for all i.

Proof. It is more convenient to work with the polynomials $\widetilde{\mathfrak{G}}_\lambda$. Now recall (2.5) and (2.6); in fact, we need to replace π_u in (2.6) with the following modified version $\widetilde{\pi}_u$ in order to compute $\widetilde{\mathfrak{G}}_\lambda$:

$$\widetilde{\pi}_u\, f(x_1, \ldots, x_k) = \frac{1}{\Delta}\, A\left(f(x_1, \ldots, x_k) \prod_{i=1}^{k} (1 + x_i)^{i-1} \right) .$$

It follows by expanding the product $\prod_{i=1}^{k}(1+x_i)^{i-1}$, and by using the definition of a Schur polynomial s_λ as $\partial_u\, x^{\lambda+\delta}$, that the coefficient $g_{\lambda\mu}$ of an arbitrary Schur polynomial s_μ in $\widetilde{\mathfrak{G}}_\lambda$ is the determinant

$$\det\left(\binom{i-1}{\mu_j - j - \lambda_i + i} \right)_{1 \le i,j \le k} .$$

By the Gessel-Viennot method (see [9]), this is precisely the number of k-tuples of non-intersecting lattice paths with steps $(1,0)$ and $(0,1)$, such that the i-th path joins the points $A_i(\lambda_i - i + 1, -\lambda_i)$ and $B_i(\mu_i - i + 1, i - 1 - \mu_i)$. Furthermore, these paths are in bijection with semistandard Young tableaux of shape μ/λ whose entries in the i-th row are integers between $-\lambda_i$ and $i - 1 - \mu_i$ (inclusive); here we use the numbering of the rows as they appear in the Young diagram of μ. By adding to every entry the index of its column, we obtain a bijection with the tableaux in the statement of the theorem. It is clear from this description that $g_{\lambda\mu} = 0$ unless $\lambda \subseteq \mu \subseteq \widehat{\lambda}$. On the other hand, if μ is in this interval, then we can always find a tableau counted by $g_{\lambda\mu}$ simply by filling in the boxes of μ/λ in row i from right to left with $i-1, i-2, \ldots$. □

The above theorem suggests that the expansion of an arbitrary Grothendieck polynomial in the basis of Schubert polynomials, which is investigated in [16], involves Schubert polynomials corresponding to permutations in a certain interval in Bruhat order.

We now give a recurrence relation for the coefficients $g_{\lambda\mu}$ involving more familiar combinatorial quantities. For $\mu \supseteq \lambda$, we let $\rho = \rho(\lambda, \mu)$ and $\tau = \tau(\lambda, \mu)$ be the partitions of smallest weight (identified with the corresponding Young diagrams) satisfying $\tau/\rho = \mu/\lambda$. We also denote by $r = r(\lambda, \mu)$ the number of rows above the first row of μ/λ, and by λ' the conjugate of λ, as usual. The recurrence relation below expresses $g_{\lambda\mu}$ in terms of $g_{\rho\nu}$ with $l(\nu) < l(\mu)$ and $l(\nu') < l(\mu')$; also note that $l(\rho) < l(\lambda)$ and $l(\rho') < l(\lambda')$, since the first parts of λ and μ are equal. The recurrence relation also involves the number of semistandard Young tableaux of certain skew shapes τ'/ν' with entries between 1 and r;

recall that the latter number is also the specialization of the skew Schur function $s_{\tau'/\nu'}$ obtained by setting the first r variables equal to 1, whence the notation $s_{\tau'/\nu'}(1^r)$.

Proposition 2.9. *We have*

$$g_{\lambda\mu} = \sum_{\rho\subseteq\nu\subseteq\widehat{\rho}\wedge\tau} g_{\rho\nu}\, s_{\tau'/\nu'}(1^r)\,,$$

where $\wedge$ denotes the meet in the Young lattice.

Proof. Every tableau enumerated by $g_{\lambda\mu}$ (as specified by Theorem 2.8) can be decomposed into two row and column-strict tableaux, one of shape ν/ρ with entries in row i between 1 and $i-r-1$, and the other of shape τ/ν with entries in row i between $i-r$ and $i-1$. By adding $r-i+1$ to the entries in row i of the second tableau, we obtain a row-strict tableau of shape τ/ν with entries between 1 and r. Furthermore, this correspondence is a bijection, whence the recurrence relation follows. $\square$

Note that a similar recurrence relation was given in [15] for the action of Steenrod operations on the cohomology of Grassmannians. The proof, however, was more difficult and not combinatorial, since there was no simple combinatorial interpretation for the corresponding coefficients.

Corollary 2.10. *If μ/λ is not a skew shape, then $g_{\lambda\mu}$ is the number of semistandard Young tableaux of shape μ'/λ' with entries between 1 and $r(\lambda,\mu)$.*

Note that the recursive computation in Proposition 2.9 always ends by using Corollary 2.10. Also note that the number of semistandard Young tableaux of a nonskew shape is easy to compute (see e.g. [18] p. 45).

Finally, let us recall another combinatorial interpretation of the coefficients $g_{\lambda\mu}$, which appears in [5], and which is valid in fact for every stable Grothendieck polynomial. In order to state it, we need to consider the modified version $\widetilde{H}_n(0)$ of the degenerate Hecke algebra $H_n(0)$, which is obtained by replacing the first relation in (2.3) with $u_i^2 = u_i$. We also consider semistandard Young tableaux with entries between 1 and $n-1$. The column word of such a tableau is obtained by reading its columns bottom-up and left-to-right. In fact, for the purposes of our next result, we usually need to read u_i instead of i, so the corresponding word will in fact be an element of $\widetilde{H}_n(0)$. The two notions of column word will be used interchangeably. Also recall that the row word of a tableau is obtained by reading its rows left-to-right and bottom-up.

Theorem 2.11. [5] *The coefficient $g_{\lambda\mu}$ is the number of semistandard Young tableaux of shape μ' and entries between 1 and $n-1$, whose column word is equal to $w(\lambda)$ in $\widetilde{H}_n(0)$.*

The tableaux in the above theorem are difficult to enumerate. Hence it makes sense to establish a bijection between the tableaux in Theorems 2.8 and 2.11. We do this below.

Consider the partitions $\lambda\subseteq\mu$, and the row and column strict tableau T_0 of shape λ' corresponding to the nilplactic class of reduced words for the Grassmannian permutation $w(\lambda)$ (see [4]). This means that both the row and column words of T_0 are reduced words for $w(\lambda)$. Consider also an arbitrary tableau T of shape μ/λ and the type specified in Theorem 2.8, and let T' be its transpose (of shape μ'/λ'). We use T' to construct from

T_0 a tableau T_1 of shape μ' of the type specified in Theorem 2.11. Assuming that the entries in the r-th row of T' are $i_1 < \ldots < i_s$, we consider the sequence $j_1 \leq \ldots \leq j_s$ with $j_l := i_l - l + 1$, and let m_k denote the number of times k appears in it. The r-th row of T_1 is then obtained from the r-th row of T_0 by inserting m_k extra copies of its k-th entry for every k. For instance, let $\lambda = (4,4,3,1,1)$, $\mu = (4,4,4,4,3)$, $w(\lambda) = (2,3,6,8,9,1,4,5,7)$. The tableau T_0, together with an example of a tableau T' and the associated tableau T_1, are presented below.

12345	$\times\times\times\ \ \times\times$	12345
456	$\times\times\times\ 1\ 2$	44456
567	$\times\times\times\ 2\ 3$	56667
78	$\times\times 1\ \ 3$	7788

Theorem 2.12. *The map $T \mapsto T_1$ defined above is a bijection between the tableaux in Theorems 2.8 and 2.11 corresponding to the partitions $\lambda \subseteq \mu$.*

Proof. For the purposes of this proof, it is useful to give another description of the algorithm for constructing T_1 given T. Assume, as above, that the entries in the r-th row of T' are $i_1 < \ldots < i_s$. Then in order to obtain the r-th row of T_1 from the r-th row of T_0, we first insert an extra copy of the i_1-th entry, then an extra copy of the i_2-th entry in the row obtained *after* the first step etc.

Let us first note that the row restrictions on T precisely say that every i_l is less or equal to the index of the previous column, which means that the algorithm is well defined. Now let us consider two consecutive rows in T_0, say rows $r-1$ and r. It is easy to check that the entries in row $r-1$ of T_0 are the consecutive integers between $\lambda_1' - \lambda_{r-1}' + r - 1$ and $\lambda_1' + r - 2$, while the entries in the r-th row of T_0 are the consecutive integers between $\lambda_1' - \lambda_r' + r$ and $\lambda_1' + r - 1$ (all we need to check is that the row word of the tableau with these entries is a reduced word for $w(\lambda)$). We insert extra copies of certain entries in these two rows in parallel, according to the entries in rows $r-1$ and r of T', which are scanned columnwise (from left to right). Let $\widetilde{T}$ denote the intermediate array. We want to show that the following property holds at every step:

$$\widetilde{T}_{r,j} > \widetilde{T}_{r-1,j} \ \text{ and } \ (\,\widetilde{T}_{r,j} = \widetilde{T}_{r,j-1} \ \Rightarrow\ \widetilde{T}_{r,j} - \widetilde{T}_{r-1,j-1} \geq 2\,) \tag{2.13}$$

for every j such that the corresponding entries of $\widetilde{T}$ exist. This is clearly true until the length of the r-th row of $\widetilde{T}$ becomes equal to the length of row $r-1$; indeed, we only have to use $(T_0)_{r,\lambda_r'} = (T_0)_{r-1,\lambda_{r-1}'} + 1$, and the fact that the entries in rows $r-1$ and r of T_0 are consecutive. We now start to insert extra copies of entries in rows $r-1$ and r of $\widetilde{T}$ alternatively. Assume that (2.13) holds until we insert extra copies of $\widetilde{T}_{r-1,p}$ and $\widetilde{T}_{r,q}$, where $p < q$ are entries in the same column of T' (which is column strict). The column strictness of $\widetilde{T}$ after the insertion is a simple consequence of the fact that $p < q$. So we only need to show that we cannot have $\widetilde{T}_{r-1,q} = \widetilde{T}_{r-1,q+1}$; but this is a consequence of the fact that T' is row strict, which means that before repeating the p-th entry in row $r-1$ of $\widetilde{T}$, the entries in positions $p, p+1, \ldots$ are strictly increasing. Property (2.13) clearly continues to hold after we finished scanning the entries in the r-th row of T'.

In particular, we have shown that T_1 is column strict. We will now use the second part of (2.13) to show that the column word of T_1 is equal to $w(\lambda)$ when evaluated in $\widetilde{H}_n(0)$.

We mentioned above that the column word of T_0 is a reduced word for $w(\lambda)$. Hence, it suffices to define a sequence of transformations of T_1, which preserve the evaluation of the column word in $\widetilde{H}_n(0)$, such that the output of the process is T_0. These transformations are as follows: we scan the rows of T_1 from right to left and from bottom to top, and remove the current entry (shifting the rest of that row by one position to the left) if it appears at least one more time in that row. Let $\widetilde{T}$ denote the tableau obtained at an intermediate step, and assume we are removing the repeated entry in position (r,p). Since row $r+1$ of $\widetilde{T}$ (if it exists) has no repeated entries, we have

$$\widetilde{T}_{r+1,j}-\widetilde{T}_{r,j}\geq 2\,,\quad\text{for all } j\geq p\,. \tag{2.14}$$

On the other hand, the second part of (2.13) and the fact that there are no more repeated entries in the r-th row of $\widetilde{T}$ after the p-th one, imply

$$\widetilde{T}_{r,j}-\widetilde{T}_{r-1,j-1}\geq 2\,,\quad\text{for all } j\geq p\,. \tag{2.15}$$

Now observe that shifting an entry of $\widetilde{T}$ from one column to the previous one, corresponds, in terms of the column word, to moving that entry past the entries below it in the same column, and then past the entries above it in the previous column. Combining this observation with (2.14) and (2.15), shows that the column words of $\widetilde{T}$ and of the tableau obtained from it by removing the entry $\widetilde{T}_{r,p}$ and shifting the entries following it are equal in $\widetilde{H}_n(0)$. Note that we do not need the third relation in (2.3) to deduce this equality.

Now let us observe that the map $T\mapsto T_1$ between the tableaux in Theorems 2.8 and 2.11 corresponding to some partitions $\lambda\subseteq\mu$ is injective; indeed, since T' is row strict, there is a unique way to reconstruct it. Since the two sets of tableaux have the same cardinality by Theorem 2.11, it follows that the above map is a bijection. □

We conclude this section with the study of the transition matrix from Schur polynomials to Grothendieck polynomials.

Theorem 2.16. *We have*

$$s_\lambda=\sum_{\substack{\mu\supseteq\lambda\\ l(\mu)\leq k,\,\mu_1=\lambda_1}} f_{\lambda\mu}\,\mathfrak{G}_\mu\,,$$

where $f_{\lambda\mu}$ is the number of semistandard Young tableaux of shape μ/λ with entries in the i-th row (counted from the top row of λ and μ) restricted to $1,2,\ldots,i-1$ for all i.

Proof. It suffices to show that if λ and ν are fixed partitions with λ strictly contained in ν and $\lambda_1=\nu_1$, then we have

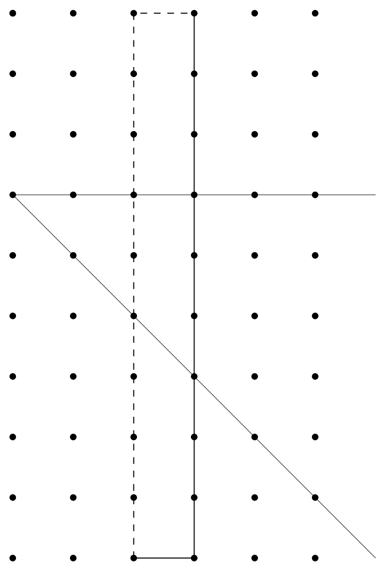
$$\sum_{\lambda\subseteq\mu\subseteq\nu}(-1)^{|\mu|-|\lambda|}g_{\lambda\mu}f_{\mu\nu}=0\,. \tag{2.17}$$

We recall our convention of appending extra parts equal to 0 to a partition, such that it has precisely k parts. By the Gessel-Viennot theory, the left-hand side of (2.17) can be interpreted as a sum over k-tuples of nonintersecting lattice paths $(p_1,\ldots,p_k)$ with steps $(1,0)$ and $(0,1)$, where p_i joins the point $A_i(\lambda_i-i+1,-\lambda_i)$ to the point $C_i(\nu_i-i+1,k+i-1)$; we also impose the condition that the subpath lying between the lines $y=-x$ and $y=k+1$ consists entirely of vertical steps (indeed, we need the i-th path to intersect the line $y=-x$ in a point $B_i(\mu_i-i+1, i-1-\mu_i)$, and the line $y=k+1$ in a point $B_i'(\mu_i-i+1, k+1)$). Every k-tuple of paths makes a contribution of 1 or -1 to the left-hand side of (2.17), depending on the total number of horizontal steps below the line $y=-x$ being even or odd, respectively. We now define a sign-reversing involution on the above k-tuples of paths, which will prove (2.17).

Assuming that $\lambda_i < \nu_i$, and that p_i is a path between A_i and C_i as above, we define a new path $S(p_i)$ between the same points in the following way. First note that the number of steps of p_i below the line $y=-x$ is one more than the number of its vertical steps above the line $y=k+1$; furthermore, every path has at least a horizontal step. Hence, precisely one of the following two cases may arise.

Case 1: The path p_i contains a subpath passing through the points $(-u-1, u-\delta)$, $(-u, u-\delta)$, $(-u, u)$, $(-u, k+1)$, and $(-u, k+1+\delta)$. Then we define $S(p_i)$ by replacing this subpath with the (unique) one passing through $(-u-1, k+1+\delta)$. This case corresponds to replacing the continuous path with the dotted one in the figure below.

Case 2: The path p_i contains a subpath passing through the points $(-u, u-\delta-1)$, $(-u, u)$, $(-u, k+1)$, $(-u, k+1+\delta)$, and $(-u+1, k+1+\delta)$. Then we define $S(p_i)$ by replacing this subpath with the (unique) one passing through $(-u+1, u-\delta-1)$. This case corresponds to replacing the dotted path with the continuous one in the figure below.



It is obvious that $S(S(p_i)) = p_i$, and that the parity of the number of horizontal steps below the line $y=-x$ in p_i and $S(p_i)$ is different. Now consider the k-tuple $(p_1, \ldots, p_k)$, and find the largest $l \geq 2$ such that p_l does not consist entirely of vertical steps and $S(p_l)$ does not intersect p_{l-1} (note that if p_{l-1} consists entirely of vertical steps, then the second condition is automatically satisfied). We claim that the map $(p_1, \ldots, p_k) \mapsto (p_1, \ldots, p_{l-1}, S(p_l), p_{l+1}, \ldots, p_k)$ is a sign-reversing involution on the set of k-tuples of lattice paths considered above. First note that $S(p_l)$ does not intersect p_{l+1} either, if $l < k$. Indeed, if p_{l+1} consists entirely of vertical steps, then this is clear. Otherwise, since $S(p_{l+1})$ intersects p_l, it follows that $S(p_{l+1})$ is obtained as in Case 2 above. Using the same assumptions for p_{l+1} as for p_i in Case 2, we deduce that p_l must have a subpath passing through $(-u+1, u-\delta-1)$. The only interesting case to check is when $S(p_l)$ is obtained as in Case 1. Clearly, only the portion of p_l between the lines $y = u - \delta - 1$ and $y = k + \delta$ is affected, whence $S(p_l)$ does not intersect p_{l+1}. Secondly, we note that if $S(p_{l+1})$ intersects p_l, then $S(p_{l+1})$ intersects $S(p_l)$ too, because the point $(-u+1, u-\delta-1)$ lies on both p_l and $S(p_l)$. Hence the map defined above is an involution, and it is clearly sign-reversing, which concludes the proof. □

3. Pieri Formulas for Grothendieck Polynomials Corresponding to Grassmannian Permutations

In this section we present Pieri formulas for Grothendieck polynomials corresponding to Grassmannian permutations, that is a rule for expanding the products $\mathfrak{G}_\lambda\,\mathfrak{G}_{(1^n)}$ and $\mathfrak{G}_\lambda\,\mathfrak{G}_{(n)}$ in the basis of Grothendieck polynomials. We note that the Pieri formula in [8] for arbitrary Grothendieck polynomials is *not* a polynomial identity, being valid only up to a certain ideal, and uses the variables $a_i = (1-x_i)^{-1}$ instead of x_i. Our Pieri formulas are direct generalizations of the classical Pieri formulas for Schur functions. We let $c(\mu/\lambda)$ and $r(\mu/\lambda)$ denote the number of columns and rows of the skew diagram μ/λ.

Theorem 3.1. *We have that*

$$\mathfrak{G}_\lambda\,\mathfrak{G}_{(1^n)} = \sum_\mu (-1)^{|\mu|-|\lambda|} \binom{c(\mu/\lambda)-1}{|\mu/\lambda|-n} \mathfrak{G}_\mu\,,$$

where the number of variables is $k \geq \max(l(\lambda), n)$, and the summation ranges over all partitions μ with $l(\mu) \leq k$ obtained by adding a vertical strip *to λ (of weight at least n).*

Proof. It is more convenient to work with the polynomials $\widetilde{\mathfrak{G}}_\lambda$. We first show that

$$\widetilde{\mathfrak{G}}_\lambda\, e_n = \sum_\nu (-1)^{|\overline{\lambda+\nu}|-|\lambda+\nu|}\, \widetilde{\mathfrak{G}}_{\overline{\lambda+\nu}}\,, \tag{3.2}$$

where the summation ranges over all binary sequences ν of length k and weight $|\nu| := \sum \nu_i$ equal to n, and $\overline{\lambda+\nu}$ is the smallest partition containing $\lambda+\nu$. Let us first note that

$$A\left(x^{\lambda+\delta}\prod_{i=1}^k (1+x_i)^{i-1}\right)\left(\sum_\nu x^\nu\right) = \sum_\nu A\left(x^{\lambda+\delta+\nu}\prod_{i=1}^k (1+x_i)^{i-1}\right)\,,$$

where the summations over ν involve the same binary sequences as above. Hence, in order to prove (3.2), it suffices to show

$$A\left(x^{\lambda+\delta+\nu}\prod_{i=1}^k (1+x_i)^{i-1}\right) = (-1)^{|\overline{\lambda+\nu}|-|\lambda+\nu|}\, A\left(x^{\overline{\lambda+\nu}+\delta}\prod_{i=1}^k (1+x_i)^{i-1}\right)\,. \tag{3.3}$$

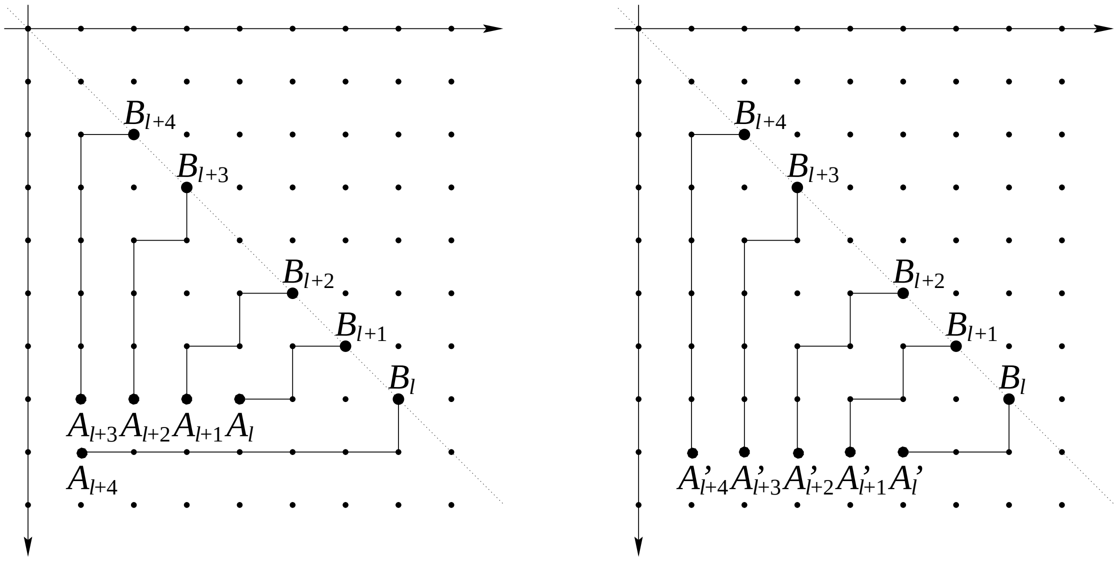
This follows by an argument involving nonintersecting lattice paths. Indeed, if ρ is a fixed partition, we have seen in the proof of Theorem 2.8 that the coefficient of the Schur polynomial s_ρ (in k variables) in the left-hand side of (3.3) is a certain binomial determinant. By Lindström's theorem (see [9]), this determinant can be expressed in terms of k-tuples of nonintersecting lattice paths with steps $(1,0)$ and $(0,1)$, such that the i-th path joins the point $A_i(\lambda_i+\nu_i-i+1, -\lambda_i-\nu_i)$ to some point $B_{j(i)}(\rho_{j(i)}-j(i)+1, j(i)-1-\rho_{j(i)})$; more precisely, every such k-tuple gives a contribution of 1 or -1 to the coefficient of s_ρ, according to the sign of the permutation $i \mapsto j(i)$. Now let us consider the unique maximal partition of the set $\{1, \ldots, k\}$ (with respect to the refinement order) into subsets of consecutive integers $\{m_j + 1, \ldots, m_{j+1}\}$ (we set $m_0 := 0$) such that the following conditions are simultaneously satisfied for all $j = 0, 1, \ldots$:

1. $\lambda_{m_j+1} = \ldots = \lambda_{m_{j+1}}$;
2. if $\nu_{m_{j+1}} = 1$ then $\nu_{m_j+1} = \ldots = \nu_{m_{j+1}-1} = 0$, otherwise $m_{j+1} = m_j + 1$.

Define points A_i' for $1 \leq i \leq k$ by

$$x(A_i') = \begin{cases} x(A_i) + 1 & \text{if } (\exists)\, j \,:\, m_j + 1 \leq i < m_{j+1} \\ x(A_i) & \text{otherwise}, \end{cases}$$
$$y(A_i') = \begin{cases} y(A_i) - 1 & \text{if } (\exists)\, j \,:\, m_j + 1 \leq i < m_{j+1} \\ y(A_i) & \text{otherwise}, \end{cases}$$

where $x(P)$ and $y(P)$ denote the x and y-coordinates of the point P. The x-coordinates of the points A_i' are strictly decreasing (as i increases), and their y-coordinates are weakly increasing. Indeed, there is a unique k-tuple μ satisfying $x(A_i') = \mu_i - i + 1$ and $y(A_i') = -\mu_i$ for all $l = 1, \ldots, k$, and that is precisely the *partition* $\overline{\lambda + \nu}$. We claim that there is a one-to-one correspondence between the k-tuples of nonintersecting lattice paths joining the points A_i to B_j $(1 \leq i, j \leq k)$, and the k-tuples of nonintersecting lattice paths joining the points A_i' to B_j. The crucial observation is that if $m_j + 1 < m_{j+1}$, then the first $m_{j+1} - m_j - 1$ steps of the path starting at $A_{m_{j+1}}$ are necessarily $(1, 0)$; after these steps, the path reaches the point A'_{m_j+1}. On the other hand, under the same assumption, the first step of a path starting at A_i', with $m_j + 1 < i \leq m_{j+1}$, is necessarily $(0, 1)$; after this step, the path reaches the point A_{i-1}. This bijection can be best understood by examining the two figures below, where $l = m_j + 1$ and $m_{j+1} = m_j + 5$. Since the only k-tuples of nonintersecting lattice paths joining A_i' to B_j are those in which A_i is joined to B_i, it follows that the only k-tuples of nonintersecting lattice paths joining A_i to B_j are those in which A_i is joined to B_{i+1} if $m_j + 1 \leq i < m_{j+1}$ for some j, and to B_{m_j+1} if $i = m_{j+1}$. The sign of the corresponding permutation of the set $\{1, \ldots, k\}$ is $(-1)^{\sum m_{j+1} - m_j - 1} = (-1)^{|\overline{\lambda+\nu}| - |\lambda+\nu|}$. This completes the proof of (3.2).



Let us now deduce the Pieri formula from (3.2). According to Theorem 2.8, we have

$$\widetilde{\mathfrak{G}}_{(1^n)} = \sum_{i=n}^{k} \binom{i-1}{i-n} e_i \,.$$

Combining this formula with (3.2), we deduce that $\widetilde{\mathfrak{G}}_\lambda \, \widetilde{\mathfrak{G}}_{(1^n)}$ can be written as a sum of $\widetilde{\mathfrak{G}}_\mu$ with μ/λ a vertical strip of weight $p \geq n$. Furthermore, it is not difficult to see that the corresponding coefficient is

$$\sum_{i=0}^{\min(p-n,p-c(\mu/\lambda))} (-1)^i \binom{p-1-i}{p-n-i} \binom{p-c(\mu/\lambda)}{i} \,.$$

On the other hand, this is precisely the coefficient of x^{p-n} in the expansion of

$$(1-x)^{p-c(\mu/\lambda)} \, (1-x)^{-n} = (1-x)^{-(n+c(\mu/\lambda)-p)} \,.$$

In other words, it is 0 if $n + c(\mu/\lambda) \leq p$, and it is equal to $\binom{c(\mu/\lambda)-1}{p-n}$ otherwise. ∎

Theorem 3.4. *We have that*

$$\mathfrak{G}_\lambda \, \mathfrak{G}_{(n)} = \sum_\mu (-1)^{|\mu|-|\lambda|} \binom{r(\mu/\lambda)-1}{|\mu/\lambda|-n} \mathfrak{G}_\mu \,,$$

where the number of variables is $k \geq l(\lambda)$, and the summation ranges over all partitions μ with $l(\mu) \leq k$ obtained by adding a horizontal strip *to λ (of weight at least n).*

Proof. Again, it is more convenient to work with the polynomials $\widetilde{\mathfrak{G}}_\lambda$. We first show that

$$\widetilde{\mathfrak{G}}_\lambda \, h_n = \sum_\nu (-1)^{|\nu|-|\lambda|-n} \, \widetilde{\mathfrak{G}}_\nu \,, \tag{3.5}$$

where the summation ranges over all partitions ν (identified with their Young diagrams) containing λ and satisfying $l(\nu) \leq k$ and $c(\nu/\lambda) = n$. Let us first rephrase (3.2) and (3.5) in terms of certain tableaux T of shape $\mathrm{sh}(T) = \nu/\lambda$ with entries 0 and 1, for which the bottom element in every column is 1. We denote by $\alpha_i(T)$ the number of entries equal to i in T, and by $\lambda + (1^k)$ the partition obtained by incrementing all parts of λ by 1. With this notation, (3.2) and (3.5) can be written

$$\widetilde{\mathfrak{G}}_\lambda \, e_n = \sum_{T\,:\,\mathrm{sh}(T)=\nu/\lambda} (-1)^{\alpha_0(T)} \, \widetilde{\mathfrak{G}}_\nu \,, \tag{3.6}$$

$$\widetilde{\mathfrak{G}}_\lambda \, h_n = \sum_{T\,:\,\mathrm{sh}(T)=\nu/\lambda} (-1)^{\alpha_0(T)} \, \widetilde{\mathfrak{G}}_\nu \,; \tag{3.7}$$

the first summation ranges over those tableaux T mentioned above for which $\nu \subseteq \lambda + (1^k)$ and $\alpha_1(T) = n$, while the second summation ranges over those T which satisfy $\alpha_1(T) = n$, and which have no two 1's in the same column.

We now prove (3.7) by induction on n using the well-known fact

$$h_n = \sum_{i=1}^{n} (-1)^{i-1} e_i h_{n-i} \,.$$

Induction starts at $n = 1$ by (3.6), so assume (3.7) holds for all $i < n$. Combining the formula for h_n with (3.6) and (3.7) (the latter for $i < n$), we can express $\widetilde{\mathfrak{G}}_\lambda \, h_n$ as a sum over certain tableaux $T = T_1 \cup T_2$ of shape ν/λ, where T_1 and T_2 are of the types mentioned above for the tableaux in (3.6) and (3.7), respectively. Fix such a tableau T, and let us evaluate its contribution to the coefficient of $\widetilde{\mathfrak{G}}_\nu$ in the above sum. This amounts to investigating all possible ways of splitting T into T_1 and T_2. First assume that there is a column of T containing at least two entries equal to 1. All columns of T which do not lie entirely in the vertical strip $(\lambda + (1^k))/\lambda$ are partitioned in a unique way between T_1 and T_2. As far as the other columns are concerned, we have the following choice: we can place the bottom entry in each of them in either T_1 or T_2. Hence, if there are m such columns (if $n > 0$ then $m > 0$), then the contribution of T to the coefficient of $\widetilde{\mathfrak{G}}_\nu$ is $\pm \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} \right) = 0$. The case when all the columns of T have only their bottom entry equal to 1 is similar. Indeed, we only have to take care not to place in T_2 all bottom entries of the columns contained in $(\lambda + (1^k))/\lambda$. Hence, such a tableau T contributes $(-1)^{\alpha_0(T)}$ to the coefficient of $\widetilde{\mathfrak{G}}_\nu$.

We can now use (3.5) to prove the theorem. According to Theorem 2.8 and a well-known formula for hook Schur functions (see, e.g., [18] p. 47), we have

$$\widetilde{\mathfrak{G}}_{(n)} = \sum_{i=0}^{k-1} s_{(n,1^i)} = \sum_{\substack{i,j \geq 0 \\ i+j \leq k}} (-1)^i h_{n+i} e_j \,.$$

Combining this formula with (3.2) and (3.5), we can express $\widetilde{\mathfrak{G}}_\lambda \, \widetilde{\mathfrak{G}}_{(n)}$ as a sum over partitions $\lambda \subseteq \nu \subseteq \mu$, where $\nu \subseteq \lambda + (1^k)$ and $n \leq c(\mu/\nu) \leq n + k$. Fix a partition μ containing λ, and let us compute the coefficient of $\widetilde{\mathfrak{G}}_\mu$ in the above sum. First assume that μ/λ is not a horizontal strip, and let ν be as above. If $m := r(\nu/\lambda) - c(\nu/\lambda) > 0$, then the contribution of ν to the coefficient of $\widetilde{\mathfrak{G}}_\mu$ is $\pm \left(\sum_{i=0}^{m} (-1)^i \binom{m}{i} \right) = 0$. On the other hand, since μ/λ is not a horizontal strip, there is a row $p \leq k$ such that column $\lambda_p + 1$ of μ/λ contains at least two boxes, and either $p = 1$ or $\lambda_p < \lambda_{p-1}$. Hence, we can pair all partitions ν with $r(\nu/\lambda) = c(\nu/\lambda)$ in the following way: if ν is such a partition with

$\nu_p = \lambda_p$, we pair it with $\nu+\varepsilon_p$; here ε_p is the binary sequence of length k having a unique entry equal to 1 in its p-th position. The crucial fact is that $c(\mu/\nu) = c(\mu/(\nu+\varepsilon_p))$, which means that $\nu+\varepsilon_p$ satisfies the same properties (stated above) as ν. If ν and $\nu+\varepsilon_p$ are paired, then their contributions to the coefficient of $\widetilde{\mathfrak{G}}_\mu$ cancel out.

Finally, we have to deal with the case when μ/λ is a horizontal strip. In each row i of μ/λ we have a choice: we either let $\nu_i = \lambda_i$ or we let $\nu_i = \lambda_i + 1$. Hence the coefficient of $\widetilde{\mathfrak{G}}_\mu$ in the expansion of $\widetilde{\mathfrak{G}}_\lambda \widetilde{\mathfrak{G}}_{(n)}$ is equal to

$$\sum_{i=0}^{|\mu/\lambda|-n} (-1)^{|\mu/\lambda|-n-i} \binom{r(\mu/\lambda)}{i} = \binom{r(\mu/\lambda)-1}{|\mu/\lambda|-n}.$$

The latter is a well-known binomial identity, which can be easily proved by induction. □

We note that it is not possible to deduce the formula in Theorem 3.4 from the formula in Theorem 3.1 directly. Indeed, the involution on the ring of symmetric polynomials in k variables sending $\mathfrak{G}_\lambda$ to $\mathfrak{G}_{\lambda'}$ is not a ring homomorphism, although the two formulas above suggest that it might be.

Theorem 3.4 allows us to define an "augmented" Kostka matrix. Recall that the classical Kostka matrix is the transition matrix between complete homogeneous symmetric functions and Schur functions; its entry $K_{\mu\lambda}$ represents the number of semistandard Young tableaux of shape μ containing λ_r entries equal to r for every r. Given a sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\mu| \geq |\alpha| := \sum \alpha_r$, we define $\widehat{K}_{\mu\alpha}$ to be the number of semistandard Young tableaux T of shape μ with entries $1 < \overline{1} < 2 < \overline{2} < \ldots < m < \overline{m}$ satisfying the following conditions for $r = 1, \ldots, m$:

1. the number of entries equal to r is α_r;
2. there is at most one entry equal to $\overline{r}$ in a row;
3. the first (possibly unique) row containing an entry equal to r contains no entries equal to $\overline{r}$.

Hence we have the following Corollary of Theorem 3.4.

Corollary 3.8. *For any sequence of positive integers $\alpha := (\alpha_1, \ldots, \alpha_m)$, we have that*

$$\mathfrak{G}_{(\alpha_1)} \ldots \mathfrak{G}_{(\alpha_m)} = \sum_{\substack{\mu\,:\,l(\mu)\leq k\\ |\mu|\geq|\alpha|}} (-1)^{|\mu|-|\alpha|} \widehat{K}_{\mu\alpha} \mathfrak{G}_\mu .$$

We conclude with two open problems. First, there is the problem of generalizing the well-known Littlewood-Richardson rule to the polynomials $\mathfrak{G}_\lambda$. This would lead to a complete combinatorial description of the multiplication of Schubert classes in $K^0(Gr_k(\mathbb{C}^n))$. Secondly, there is the problem of finding a representation-theoretical interpretation for the polynomials $\widetilde{\mathfrak{G}}_\lambda$ similar to the one for Schur functions. For instance, we might ask for the construction of polynomial representations of the general linear group whose characters are $\widetilde{\mathfrak{G}}_\lambda$.

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