SOME APPLICATIONS OF INCIDENCE HOPF ALGEBRAS TO FORMAL GROUP THEORY AND ALGEBRAIC TOPOLOGY

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1. Introduction

Hopf algebras are achieving prominence in combinatorics through the influence of G.-C. Rota and his school, who developed the theory of incidence Hopf algebras (see [7], [15], [16]). The aim of this paper is to show that incidence Hopf algebras of partition lattices provide an efficient combinatorial framework for formal group theory and algebraic topology. We start by showing that the universal Hurwitz group law (respectively the universal formal group law) are generating functions for certain leaf-labelled trees (repectively plane trees with colored leaves); the proof uses the combinatorial expression for Lagrange inversion in terms of leaf-labelled trees, which is due to M. Haiman and W. Schmitt [5]. A formal group law identity with a combinatorial proof (similar to the one in [8]) is also presented.

The relevance of Hopf algebras and formal group theory to algebraic topology (and in particular to K-theory and bordism theory) is well-known. The relevance of the Roman-Rota umbral calculus, as an elegant and illuminating framework for computations, became clear through the work of N. Ray, F. Clarke, A. Baker et al. (see [12], [11], [2], [3]). Applications of incidence Hopf algebra techniques to algebraic topology were given recently by N. Ray and W. Schmitt. In this work, we discuss some of their applications from a different point of view, and present other examples of computations (both classical and new) which can be expressed in a very concise form using the incidence Hopf algebra framework. Such applications include: coactions of $MU_*(MU)$ and $K_*(K)$, expressing the images of the coefficients of the universal formal group law under the Hurewicz homomorphism $MU_* \to K_*(MU)$, congruences in MU_*, and combinatorial models for the dual of the Steenrod algebra.

We refer the reader to [18] for all information concerning Hopf algebras. All the prerequisites concerning formal group law theory can be found in [8] §3; we also refer to [6] for an encyclopaedic description. As far as algebraic topology is concerned, the relevant prerequisites are presented in [1] and [19]. However, this paper can be read without prior knowledge of algebraic topology, the only disadvantage being that the motivation for certain computations will be less clear.

2. Combinatorial Background

Let Π_n denote the lattice of partitions of the set $[n] := \{1, 2, \ldots, n\}$, and $I\Pi_n$ the Boolean algebra of those partitions of $[n]$ for which all the blocks are intervals (in $\mathbb{N}$). We write $\Pi_{n,k}$ and $I\Pi_{n,k}$ for the subsets of Π_n and $I\Pi_n$ consisting of partitions with k blocks.

Consider the family of posets which are isomorphic to a finite product of lattices Π_n. The *incidence Hopf algebra* of this family modulo isomorphism of posets (see [15], [16]) is the Faà di Bruno Hopf algebra $\Phi_* := \mathbb{Z}[\phi_1, \phi_2, \ldots]$, with comultiplication and counit specified by

$$\delta(\phi_n) = \sum_{\sigma \in \Pi_{n+1}} \tau^\phi(\sigma) \otimes \phi_{|\sigma|-1}\,, \qquad \varepsilon(\phi_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.1}$$

for all $n \geq 0$; here $\tau^\phi(\sigma)$ is the type of the partition σ, ϕ_n denotes the isomorphism class of the lattice Π_{n+1}, and $\phi_0 = 1$.

Now consider the family of posets which are isomorphic to a finite product of Boolean algebras $I\Pi_n$. The incidence Hopf algebra of this family modulo isomorphism of posets is the Hopf algebra $(H{\cdot}\Phi)_* := \mathbb{Z}[b_1, b_2, \ldots]$, with comultiplication specified by

$$\begin{gathered} \delta(b_n) = \sum_{\sigma \in I\Pi_{n+1}} \tau^b(\sigma) \otimes b_{|\sigma|-1} = \sum_{k \geq 1} \sum_{\substack{n_1+n_2+\cdots+n_k=n+1 \\ n_i \geq 1}} \left(\prod_{i=1}^{k} b_{n_i-1} \right) \otimes b_{k-1}\,, \\ \varepsilon(b_n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise}\,, \end{cases} \end{gathered} \tag{2.2}$$

for all $n \geq 0$; here b_n denotes the isomorphism class of the Boolean algebra $I\Pi_{n+1}$, and $b_0 = 1$. It is not difficult to check that Φ_* can be identified with a sub-Hopf algebra of $(H{\cdot}\Phi)_*$ via the monomorphism $\phi_n \mapsto (n+1)!b_n$.

Let R_* be an evenly graded commutative ring. Consider the set $\operatorname{Hom}(\Phi_*, R_*)$ of graded ring homomorphisms from Φ_* to R_*. This set is a group under the operation of convolution, which is dual to the comultiplication in Φ_*; more explicitely, we have

$$(f * g)(z) = m \circ (f \otimes g) \circ \delta\,(z) \qquad \text{and} \qquad f^{-1}(z) = f \circ S(z)\,,$$

where m denotes the multiplication in R_*, and S is the antipode of Φ_*. The map $(f \otimes g) \circ \delta$ in $\operatorname{Hom}(\Phi_*, R_* \otimes R_*)$ will be denoted by $f \circledast g$. A sequence of elements $\rho = (1 = \rho_0, \rho_1, \ldots)$ in R_* with $\rho_i \in R_{2i}$ will be called an *umbra.* Let ζ^ρ in $\operatorname{Hom}(\Phi_*, R_*)$ be the ring homomorphism specified by $\phi_i \mapsto \rho_i$, let μ^ρ be its convolution inverse, and let $\overline{\rho}$ be the umbra specified by $\zeta^{\overline{\rho}} = \mu^\rho$; in this context, if $\phi = (1 = \phi_0, \phi_1, \ldots)$, then ζ^ϕ is the identity map, and μ^ϕ is the antipode of Φ_*. The correspondence $R_* \mapsto \operatorname{Hom}(\Phi_*, R_*)$ is a covariant functor from the category of graded rings to the category of groups. For some group homomorphisms $\gamma\colon \operatorname{Hom}(\Phi_*, R_*) \to \operatorname{Hom}(\Phi_*, T_*)$, as those mentioned in

§3, there is a unique ring homomorphism $\chi\colon R_* \to T_*$ such that $\mathrm{Hom}(\chi) = \gamma$. The above discussion also applies to the group $\mathrm{Hom}((H\cdot\Phi)_*, R_*)$ of graded ring homomorphisms from $(H\cdot\Phi)_*$ to R_*.

The following is a well-known result (see [4], Theorem 5.1 and [16], Examples 14.1 and 14.2).

Theorem 2.3.

1. *The group* $\mathrm{Hom}(\Phi_*, R_*)$ *is anti-isomorphic to the group under substitution of* Hurwitz series *in* $R^*\{\{X\}\}$ *of the form*
$$\rho(X) := \sum_{i\geq 1} \rho_{i-1} X_{(i)}\,,$$
where ρ *is an umbra in* R_* *(if* R_* *has no torsion, then we can identify* $X_{(i)}$ *with* $X^i/i!$*); the anti-isomorphism is specified by* $\zeta^\rho \mapsto \rho(X)$.
2. *The group* $\mathrm{Hom}((H\cdot\Phi)_*, R_*)$ *is anti-isomorphic to the group under substitution of formal power series in* $R^*[[X]]$ *of the form*
$$r(X) := \sum_{i\geq 1} r_{i-1} X^i\,,$$
where r *is an umbra in* R_**; the anti-isomorphism is specified by* $\zeta^r \mapsto r(X)$.

We have now established an important notational convention, to which we shall adhere below: a Hurwitz series and a map in $\mathrm{Hom}(\Phi_*, R_*)$ (or a standard formal power series and a map in $\mathrm{Hom}((H\cdot\Phi)_*, R_*)$) will be associated with an umbra denoted by a Greek letter (or lower case Roman letter). The following notational convention will also be useful: if P^ϕ lies in $\Phi_*[x]$ and Q^b lies in $(H\cdot\Phi)_*\{x\}$, then P^ρ in $R_*[x]$ and Q^r in $R_*\{x\}$ are obtained from P^ϕ and Q^b via the substitutions $\phi_i \mapsto \rho_i$ and $b_i \mapsto r_i$, respectively; furthermore, $P^{(1,1,\dots)}$ will be denoted simply by P.

Let us now consider the binomial Hopf algebra $\Phi_*[x]$, and the divided power Hopf algebra $(H\cdot\Phi)_*\{x\}$ (see [8], §2 for a detailed discussion). Recall the *conjugate Bell polynomials* and the *Bell polynomials* in $\Phi_*[x]$; they can be expressed combinatorially as follows:
$$B_n^\phi(x) = x\,(\mu^\phi * \zeta^\xi)(\phi_{n-1}) \quad \text{for } n \geq 1\,, \tag{2.4}$$
where ξ is the umbra $(1, x, x^2, \dots)$ in $\Phi_*[x]$. Let us write
$$B_n^\phi(x) = \sum_{k=1}^{n} s^\phi(n,k) x^k \qquad \text{and} \qquad B_n^{\overline{\phi}}(x) = \sum_{k=1}^{n} S^\phi(n,k) x^k\,.$$
The coefficients $s^\phi(n,k)$ and $S^\phi(n,k)$ in $\Phi_{2(n-k)}$ are known as *Φ_*-Stirling numbers* of the first and second kind, respectively; $S^\phi(n,k)$ are also known as *partial Bell polynomials*. The standard notation $s(n,k)$ and $S(n,k)$ for the classical Stirling numbers of the first and second kind is consistent with our conventions.

Let us consider the umbra $x = (1, x/2, x^2/3!, \dots)$ in $(\Phi \otimes \mathbb{Q})_*[x]$, and the following polynomials in $(H\cdot\Phi)_*\{x\}$:

$$\beta_0^b(x) = 1\,, \qquad \text{and} \qquad \beta_n^b(x) := x\,(\mu^b * \zeta^x)(b_{n-1}) \quad \text{for } n \geq 1\,. \tag{2.5}$$

Since Φ_* is a sub-Hopf algebra of $(H\cdot\Phi)_*$ via the inclusion i, we have the group homomorphism $i^*\colon \operatorname{Hom}((H\cdot\Phi)_*, (\Phi \otimes \mathbb{Q})_*[x]) \to \operatorname{Hom}(\Phi_*, (\Phi \otimes \mathbb{Q})_*[x])$ dual to i, which is specified by $f \mapsto f \circ i$; more explicitely, we have that $i^*(\zeta^b) = \zeta^\phi$ and $i^*(\zeta^x) = \zeta^\xi$, whence $i^*(\mu^b) = \mu^\phi$ by taking inverses. Hence, identifying $\Phi_*[x]$ and $(H\cdot\Phi)_*\{x\}$ with the corresponding subrings of $(\Phi \otimes \mathbb{Q})_*[x]$, and using (2.4) and (2.5), we deduce that

$$B_n^\phi(x) = n!\,\beta_n^b(x)\,. \tag{2.6}$$

A generalization of this result using set systems and their automorphism groups was obtained in [9]. Writing

$$\beta_n^b(x) = \sum_{k=1}^{n} \beta_{n,k}^b \frac{x^k}{k!}\,,$$

we clearly have $n!\,\beta_{n,k}^b = k!\,s^\phi(n,k)$.

3. Applications to Formal Group Theory

Let us denote, as in [8], by $\Phi^*\{\{X,Y\}\}$ the Φ^*-algebra of Hurwitz series in two indeterminates. According to [8] §3, there are formal power series

$$F^\phi(X,Y) \in \Phi^1\{\{X,Y\}\} \quad \text{and} \quad f^b(X,Y) \in (H\cdot\Phi)^1[[X,Y]]$$

such that

$$F^\phi(\phi(X), \phi(Y)) = \phi(X+Y) \quad \text{and} \quad f^b(b(X), b(Y)) = b(X+Y)\,.$$

Let L_* be the subring of $(H\cdot\Phi)_*$ generated by the coefficients of $f^b(X,Y)$; this is the Lazard ring, which is known to be isomorphic to the complex cobordism ring MU_* (see [1]). The formal power series $F^\phi(X,Y)$ and $f^b(X,Y)$ are precisely the *universal Hurwitz group law* (over Φ_*) and the *universal formal group law* (over L_*). We use the notations

$$X +_\phi Y := F^\phi(X,Y) \quad \text{and} \quad X +_b Y := f^b(X,Y)\,,$$

which can be iterated. The coefficients of

$$\prod_{i=1}^{k} \frac{X_i^{n_i}}{n_i!} \ \text{ in } \ \sum_{i=1}^{k}{}_\phi\, X_i \quad \text{and} \quad \prod_{i=1}^{k} X_i^{n_i} \ \text{ in } \ \sum_{i=1}^{k}{}_b\, X_i$$

will be denoted by $F^\phi_{n_1,\dots,n_k}$ and $f^b_{n_1,\dots,n_k}$, respectively. The first one lies in Φ_*, while the second one lies in L_*.

In [8], we expressed $F^{\phi}_{n_1,\ldots,n_k}$ in terms of μ^{ϕ} as follows:

$$F^{\phi}_{n_1,\ldots,n_k}=\sum_{\pi\geq\sigma\in\Pi_n}\mu^{\phi}(\widehat{0},\sigma)\,\zeta^{\phi}(\sigma,\widehat{1})\,; \tag{3.1}$$

here, and throughout this section, $n:=n_1+\ldots+n_k$, and π is the partition of $[n]$ with blocks $[n_1]$, $n_1+[n_2]:=\{n_1+1,\ldots,n_1+n_2\}$, ..., $n-n_k+[n_k]$. We are going to present a similar expression (3.6) for $f^b_{n_1,\ldots,n_k}$. Several cancellations occur in these expressions. However, we are able to give a combinatorial interpretation for the coefficients of the monomials in $F^{\phi}_{n_1,\ldots,n_k}$ and $f^b_{n_1,\ldots,n_k}$ in terms of trees, by using the Haiman-Schmitt form of Lagrange inversion [5]. To do this, we need to choose other polynomial generators for Φ_* and $(H\cdot\Phi)_*$, namely $\overline{\phi}_1,\overline{\phi}_2,\ldots$, and $\overline{b}_1,\overline{b}_2,\ldots$. We consider rooted trees with n leaves of two types: rooted trees with leaves labelled $1,2,\ldots,n$, and rooted plane trees with a k-coloring of the leaves of type $(n_1,\ldots,n_k)$ (i.e. a coloring with colors $1,\ldots,k$ such that exactly n_i leaves are colored i). We also assume that no non-root vertex has degree 2 (i.e. only one descendent). The number of vertices of a tree T is denoted by $|T|$. A vertex of a tree of either type is called *peripheral* if all its descendents are leaves; the set of descendents of a peripheral vertex will be called a *peripheral class*. The type $\Omega(T')$ of a tree T' of the first type is defined as $\overline{\phi}_1^{\,i_1}\overline{\phi}_2^{\,i_2}\ldots$, where i_j is the number of vertices of T' with $j+1$ descendents; the type of a tree of the second type is defined similarly, as a monomial in $\overline{b}_1,\overline{b}_2,\ldots$. We can now state the combinatorial interpretation mentioned earlier.

Theorem 3.2. *We have that*

$$F^{\phi}_{n_1,\ldots,n_k}=\sum_{T'}(-1)^{|T'|-n}\,\Omega(T')\,, \tag{3.3}$$

$$f^{b}_{n_1,\ldots,n_k}=\sum_{T''}(-1)^{|T''|-n}\,\Omega(T'')\,; \tag{3.4}$$

the first sum ranges over those trees T' of the first type for which none of the sets of labels corresponding to a peripheral class are contained in a block of the partition π; the second sum ranges over those trees of the second type for which no peripheral class is monochromatic. Furthermore, we have that

$$F^{\phi}_{n_1,\ldots,n_k}=\sum_{k\geq 1}(-1)^k\,\widetilde{B}_{n+k-1,k}\,; \tag{3.5}$$

here $\widetilde{B}_{m,k}:=\sum_{\sigma}\tau^{\overline{\phi}}(\sigma)$, with summation ranging over those partitions $\sigma\in\Pi_{m,k}$ with no singleton blocks, for which none of the blocks are contained in a block of the partition π.

Proof. We denote by $\mathcal{T}_m$ the set of rooted trees with m leaves labelled $1,2,\ldots,m$; for $T\in\mathcal{T}_m$, we denote by $p(T)$ the partition of $[m]$ whose blocks are either peripheral classes or singletons. We write $\|\sigma\|$ for the number of non-singleton blocks of the partition σ. We also define a new partial order $\preceq$ on Π_n by insisting that $\pi \preceq \sigma$ if and only if σ is obtained from π by amalgamating only singleton blocks. With these notations, and using the expression for Lagrange inversion in terms of rooted leaf-labelled trees (see [5] Corollary 1), we have:

$$\begin{aligned}
F^{\phi}_{n_1,\dots,n_k} &= \sum_{\pi \geq \sigma \in \Pi_n} \zeta^{\overline{\phi}}(\widehat{0},\sigma)\,\mu^{\overline{\phi}}(\sigma,\widehat{1}) \\
&= \sum_{\pi \geq \sigma \in \Pi_n} \zeta^{\overline{\phi}}(\widehat{0},\sigma) \left(\sum_{T \in \mathcal{T}_{|\sigma|}} (-1)^{|T|-|\sigma|}\,\Omega(T) \right) \\
&= \sum_{\pi \geq \sigma \in \Pi_n} \; \sum_{T' \in \mathcal{T}_n : \sigma \preceq p(T')} (-1)^{|T'|-n-\|\sigma\|}\,\Omega(T') \\
&= \sum_{T' \in \mathcal{T}_n} (-1)^{|T'|-n}\,\Omega(T') \sum_{\sigma \leq \pi,\, \sigma \preceq p(T')} (-1)^{\|\sigma\|} .
\end{aligned}$$

To compute the last sum, assume that there are k (possibly $k=0$) non-singleton blocks of $p(T')$ which are contained in some block of π; then the only partitions σ satisfying $\sigma \leq \pi$ and $\sigma \preceq p(T')$ are those obtained from $p(T')$ by splitting all blocks into singletons, except some of the k blocks mentioned above. Hence the last sum is $\sum_{I \subseteq [k]} (-1)^{|I|} = \delta_{k,0}$, which proves (3.3). Formula (3.5) now follows by using the bijection between leaf-labelled rooted trees and partitions established in [5], Theorem 5.

To prove (3.4), we must first find an analogue of (3.1). We do this by recalling Theorem 5.3 in [9], which provides an expression for the *Möbius type function* μ^{ϕ}_P of a subposet P of Π_n (see [9]) when there is a group G acting on P. We choose P to be $[\widehat{0},\pi] \cup \{\widehat{1}\}$, and G to be the direct product of symmetric groups S_B for $B \in \pi$ (G acts on P in the obvious way). As in [9], we consider the poset $A(P)$ of all *preferential arrangements* (σ,λ) of $[n]$ with $\sigma \in P$ (recall that a preferential arrangement of $[n]$ is a pair (σ,λ), where $\sigma \in \Pi_n$ and λ is a bijection from $[|\sigma|]$ to σ, inducing a linear order on σ). Let $\overline{A}(P) := A(P) \setminus \{\widehat{1}\}$ and $\widehat{A}(P) := A(P) \sqcup \{\widehat{0}\}$. By insisting that $G(\widehat{0}) = \{\widehat{0}\}$, we obtain a poset action of G on $\widehat{A}(P)$, and hence an induced poset structure on the set of orbits $\widehat{A}(P)/G$. We have

$$\begin{aligned}
f^{b}_{n_1,\dots,n_k} &= \frac{F^{\phi}_{n_1,\dots,n_k}}{n_1!\dots n_k!} = -\frac{\mu^{\phi}_P(\widehat{0},\widehat{1})}{|G|} \qquad (3.6)\\
&= -\mu^{b}_{\widehat{A}(P)/G}(\widehat{0},\widehat{1}) = \sum_{[(\sigma,\lambda)] \in \overline{A}(P)/G} \mu^{b}(\widehat{0},\sigma)\,\zeta^{b}(\sigma,\widehat{1}) .
\end{aligned}$$

We can view $\overline{A}(P)/G$ as the subposet of $A(\Pi_n)$ consisting of "shuffles" of preferential arrangements of the sets $[n_1]$, $n_1+[n_2]$, ..., $n-n_k+[n_k]$, whose blocks are intervals (in $\mathbb{N}$) ordered in the natural way. If we do this, we can then establish a bijection between $A(P)/G$ and the set of pairs (ρ, c), where c is a k-coloring of $[n]$ of type $(n_1,\dots,n_k)$, and ρ is a partition of $[n]$ with monochromatic blocks which are intervals (in $\mathbb{N}$). Indeed, if (σ, λ) is the shuffle $(B_{i_1 j_1}, B_{i_2 j_2}, \dots)$ of $(B_{11}, B_{12}, \dots)$, ..., $(B_{k1}, B_{k2}, \dots)$, then the associated partition ρ is $\{[|B_{i_1 j_1}|], |B_{i_1 j_1}|+[|B_{i_2 j_2}|], \dots\}$, and all the elements in the r-th block of ρ are colored i_r. Finally, we denote by $\mathcal{P}_m$ the set of rooted plane trees with m leaves; if we label the leaves of $T \in \mathcal{P}_m$ with $1, \dots, m$, we can define $p(T)$ as before. Using (3.6) and the expression for Lagrange inversion in terms of rooted plane trees (the analogue of Corollary 1 in [5]), we have:

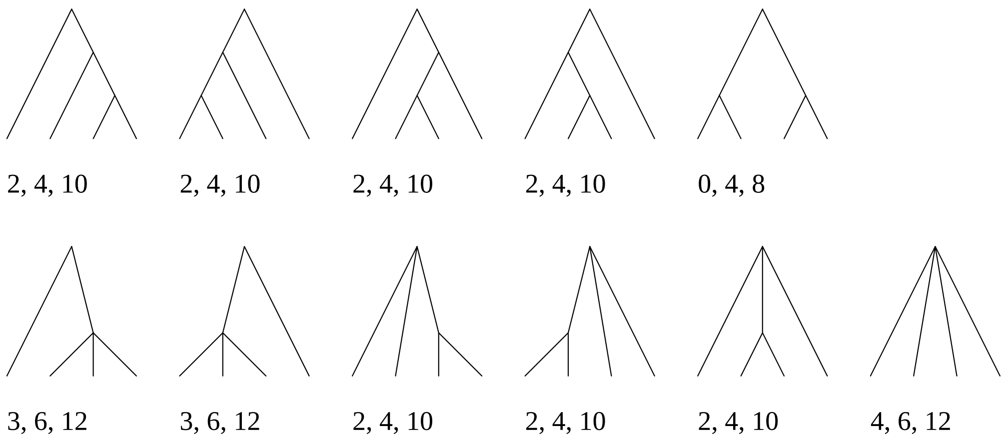
$$\begin{aligned}
f^b_{n_1,\dots,n_k} &= \sum_{(\sigma,\lambda)\in\overline{A}(P)/G} \zeta^{\overline{b}}(\widehat{0},\sigma)\,\mu^{\overline{b}}(\sigma,\widehat{1}) \\
&= \sum_{(\sigma,\lambda)\in\overline{A}(P)/G} \zeta^{\overline{b}}(\widehat{0},\sigma) \left(\sum_{T\in\mathcal{P}_{|\sigma|}} (-1)^{|T|-|\sigma|}\,\Omega(T) \right) \\
&= \sum_{(\rho,c)} \sum_{T''\in\mathcal{P}_n:\,\rho\preceq p(T'')} (-1)^{|T''|-n-||\rho||}\,\Omega(T'') \\
&= \sum_{(T'',c)} (-1)^{|T''|-n}\,\Omega(T'') \sum_{(\rho,c):\,\rho\preceq p(T'')} (-1)^{||\rho||}\,.
\end{aligned}$$

The last sum is computed as before, and (3.4) follows. □

Note that for $n_1 = \dots = n_k = 1$, the theorem is just the Haiman-Schmitt form of Lagrange inversion.

Example 3.7. *In order to express $f^b_{1,3}$, $f^b_{2,2}$, and $f^b_{1,1,2}$, we consider all the plane trees with 4 leaves, as shown below. The trees on the first line have type $\overline{b}_1^3$; those on the second line have type $\overline{b}_1\overline{b}_2$, except for the last one, whose type is $\overline{b}_3$. The three numbers corresponding to a tree represent the number of 2-colorings of type $(1,3)$, of 2-colorings of type $(2,2)$, and of 3-colorings of type $(1,1,2)$ for the leaves, which satisfy the condition in Theorem 3.2. According to the theorem, we have*

$$f^b_{1,3} = -8\overline{b}_1^3+12\overline{b}_1\overline{b}_2-4\overline{b}_3\,, \quad f^b_{2,2} = -20\overline{b}_1^3+24\overline{b}_1\overline{b}_2-6\overline{b}_3\,, \quad f^b_{1,1,2} = -48\overline{b}_1^3+54\overline{b}_1\overline{b}_2-12\overline{b}_3\,.$$



In [8], we gave a combinatorial proof of a formal group law identity using (3.1). Here we state and prove a similar identity.

Proposition 3.8. *We have that*

$$n\overline{b}_{n-1} = \sum_{\lambda \vdash n} (-1)^{l(\lambda)-1} \frac{n(l(\lambda)-1)!}{|\lambda|} f^b_\lambda ,$$

where $l(\lambda)$ is the length of the partition λ, and $|\lambda| = i_1! \ldots i_k!$ if $\lambda = (1^{i_1} \ldots k^{i_k})$.

Proof. Given $\lambda = (\lambda_1 \ldots \lambda_m)$, we write $\lambda!$ for $\lambda_1! \ldots \lambda_m!$. The type of a partition of $[n]$ (as a partition of n) will be denoted by $\lambda(\pi)$. The classical Möbius function of Π_n will be denoted by μ. We have that

$$\sum_{\pi \in \Pi_n} F^\phi_{\lambda(\pi)} \mu(\pi, \widehat{1}) = \sum_{\pi \in \Pi_n} \left(\sum_{\sigma \leq \pi} \mu^\phi(\widehat{0}, \sigma)\, \zeta^\phi(\sigma, \widehat{1}) \right) \mu(\pi, \widehat{1})$$

$$= \sum_{\sigma \in \Pi_n} \mu^\phi(\widehat{0}, \sigma)\, \zeta^\phi(\sigma, \widehat{1}) \left(\sum_{\pi \geq \sigma} \mu(\pi, \widehat{1}) \right) = \mu^\phi(\widehat{0}, \widehat{1}) = \overline{\phi}_{n-1} .$$

But $\mu(\pi, \widehat{1}) = (-1)^{|\pi|-1}(|\pi|-1)!$, and the number of partitions of $[n]$ of type λ is $n!/(\lambda!|\lambda|)$. Hence

$$n\overline{b}_{n-1} = \frac{\overline{\phi}_{n-1}}{(n-1)!} = \frac{1}{(n-1)!} \sum_{\lambda \vdash n} (-1)^{l(\lambda)-1} \frac{n!}{\lambda!|\lambda|} F^\phi_\lambda \,(l(\lambda)-1)! ,$$

which implies the identity to be proved. □

Let us note that the number $n(l(\lambda)-1)!/|\lambda|$ is an integer. Indeed, if $\lambda = (1^{i_1} \ldots k^{i_k})$, we have that

$$\frac{(i_1 + 2i_2 + \ldots + ki_k)\,(i_1 + \ldots + i_k - 1)!}{i_1! \ldots i_k!} = \binom{i_1 + \ldots + i_k - 1}{i_1 - 1,\ i_2,\ \ldots i_k}$$

$$+2\binom{i_1+\ldots+i_k-1}{i_1,\, i_2-1,\, \ldots i_k}+\ldots+k\binom{i_1+\ldots+i_k-1}{i_1,\, i_2,\, \ldots i_k-1}.$$

Hence, Proposition 3.8 provides another expression of $n\overline{b}_{n-1}$ as an integer linear combination of elements in the Lazard ring.

4. Classical Results in Algebraic Topology Restated

According to a theorem of Quillen, the ring MU_* is isomorphic to the Lazard ring, i.e. the ring generated by the coefficients of the universal formal group law. It is well-known that the Hurewicz homomorphism $MU_* \to H_*(MU) \cong \mathbb{Z}[b_1, b_2, \ldots]$ is a monomorphism, and that the elements $\phi_n := (n+1)!b_n$ all lie in MU_*. It causes no confusion to identify $(H{\cdot}\Phi)_*$ with $H_*(MU)$, and Φ_* with the subring of MU_* generated by $\phi_1, \phi_2, \ldots$, as long as we regard them only as rings, forgetting the coalgebra structure. The element $\overline{b}_n$ in $H_*(MU)$ is usually denoted by m_n, and $\overline{\phi}_n/n! = (n+1)m_n$ lies in MU_*, being the cobordism class of $\mathbb{C}P^n$, by Mischenko's theorem.

It is known that $MU_*(\mathbb{C}P_+^\infty)$ is isomorphic to the covariant bialgebra of the universal formal group law; more precisely, it is isomorphic to the free MU_*-module on generators $\beta_1^{MU}, \beta_2^{MU}, \ldots$. It was shown in [12] that the Hurewicz homomorphism $MU_*(\mathbb{C}P_+^\infty) \to (H \wedge MU)_*(\mathbb{C}P_+^\infty) \cong (H{\cdot}\Phi)_*\{x\}$ maps β_n^{MU} to $\beta_n^b(x)$.

Let us consider the *Hopf algebroid* (see [10]) $MU_*(MU) \cong MU_*[b_1^{MU}, b_2^{MU}, \ldots] \cong MU_* \otimes S_*$, where S_* is the dual of the Landweber-Novikov algebra. We note first that S_* is isomorphic to the Hopf algebra $(H{\cdot}\Phi)_*$ defined in §2. Let η_L, η_R be the left and right units, δ the comultiplication, and c the conjugation of $MU_*(MU)$. We write ϕ_n^{MU} for $(n+1)!b_n^{MU}$, ϕ_n^R for $\eta_R(\phi_n)$, and consider the umbra $b^{MU} := (1, b_1^{MU}, b_2^{MU}, \ldots)$, $\phi^{MU} := (1, \phi_1^{MU}, \phi_2^{MU}, \ldots)$, and $\phi^R := (1, \phi_1^R, \phi_2^R, \ldots)$. The right unit η_R is usually expressed via the formula

$$\overline{\phi}^R(X) = \overline{\phi}(\overline{\phi}^{MU}(X)).$$

By Theorem 2.3 (1) this is equivalent to specifying η_R as follows:

$$\eta_R: \qquad \mu^\phi \mapsto \mu^{\phi^R} = \mu^{\phi^{MU}} * \mu^\phi \qquad \text{or} \qquad \zeta^\phi \mapsto \zeta^{\phi^R} = \zeta^\phi * \zeta^{\phi^{MU}}, \tag{4.1}$$

by taking inverses. The comultiplication δ can be specified using the notation in §2 in either of following ways:

$$\delta: \qquad \zeta^{b^{MU}} \mapsto \zeta^{b^{MU}} \circledast \zeta^{b^{MU}} \qquad \text{or} \qquad \zeta^{\phi^{MU}} \mapsto \zeta^{\phi^{MU}} \circledast \zeta^{\phi^{MU}}. \tag{4.2}$$

Similarly, the conjugation c is specified by

$$c: \qquad \zeta^{b^{MU}} \mapsto \mu^{b^{MU}} \qquad \text{or} \qquad \zeta^{\phi^{MU}} \mapsto \mu^{\phi^{MU}}. \tag{4.3}$$

We will present in §5 and §6 more consistent applications, demonstrating the computational advantages of this new point of view. For now, we consider a few simple applications, demonstrating the advantages in simplifying notation and proofs. For our applications, we need the umbra $\kappa := (1, u, u^2, \dots)$ in $K_* \cong \mathbb{Z}[u, u^{-1}]$ and $\kappa^R := (1, v, v^2, \dots)$ in $K_*(K)$, where $v = \eta_R^K(u)$ (let us recall that the image of $K_*(K)$ in $K_*(K) \otimes \mathbb{Q}$ consists precisely of those finite Laurent series $f(u, v)$ satisfying $f(mt, nt) \in \mathbb{Z}[t, t^{-1}, \frac{1}{mn}]$ for all $m, n \in \mathbb{Z} \setminus \{0\}$). We also need the standard map of ring spectra $g \colon MU \to K$ representing the universal Thom class in $K^0(MU)$; the map $g_* \colon MU_* \to K_*$ is the Todd genus, mapping ϕ_n to u^n. Finally, let us recall the fact that $K_*(MU) \cong \mathbb{Z}[u, u^{-1}, b_1^K, b_2^K, \dots]$, and that $g_* \colon MU_*(MU) \to K_*(MU)$ maps b_n^{MU} to b_n^K. We will need the umbra $b^K := (1, b_1^K, b_2^K, \dots)$ and $\phi^K := (1, \phi_1^K, \phi_2^K, \dots)$, where $\phi_n^K := (n+1)! b_n^K$.

Example 4.4. Let us first check that the elements ϕ_n^R are *primitive*. Indeed, we have

$$\delta: \qquad \zeta^{\phi^R} = \zeta^\phi * \zeta^{\phi^{MU}} \mapsto \zeta^\phi * \zeta^{\phi^{MU}} \circledast \zeta^{\phi^{MU}} = \zeta^{\phi^R} \circledast \zeta^{\phi^{MU}} = 1 \otimes \zeta^\phi * \zeta^{\phi^{MU}} = 1 \otimes \zeta^{\phi^R}.$$

Example 4.5. The map $f_* \colon MU_*(\mathbb{C}P_+^\infty) \to MU_*(MU)$ induced by the inclusion $f \colon \mathbb{C}P^\infty \simeq MU(1) \hookrightarrow \Sigma^2 MU$ is specified by $\beta_n^{MU} \mapsto b_{n-1}^{MU}$. Let us identify $MU_*(\mathbb{C}P_+^\infty)$ with its image in $(H \wedge MU)_*(\mathbb{C}P_+^\infty)$, and try to determine the map f_*. It is a well-known fact in umbral calculus that

$$x^n = \sum_{\sigma \in \Pi_n} \zeta^\phi(\widehat{0}, \sigma)\, B_n^\phi(x)\,.$$

Applying the map f_*, we obtain

$$f_*(x^n) = (\zeta^\phi * \zeta^{\phi^{MU}})(\phi_{n-1}) = \phi_{n-1}^R\,. \tag{4.6}$$

Hence, the map f_* can be interpreted as *umbral substitution* at ϕ^R.

Formula (4.6) and the following commutative diagram show that the elements x^n are primitive under the coaction $MU_*(\mathbb{C}P_+^\infty) \to MU_*(MU) \otimes_{MU_*} MU_*(\mathbb{C}P_+^\infty)$.

$$\begin{array}{ccc} MU_*(\mathbb{C}P_+^\infty) & \longrightarrow & MU_*(MU) \otimes_{MU_*} MU_*(\mathbb{C}P_+^\infty) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle I \otimes f_*} \\ MU_*(MU) & \xrightarrow{\delta} & MU_*(MU) \otimes_{MU_*} MU_*(MU) \end{array} .$$

It is shown in [3] that $K_*(MU)$ is isomorphic to the direct limit

$$MU_*(\mathbb{C}P_+^\infty) \xrightarrow{\cdot x} MU_*(\mathbb{C}P_+^\infty) \xrightarrow{\cdot x} MU_*(\mathbb{C}P_+^\infty) \xrightarrow{\cdot x} \dots ,$$

where the maps are "multiplication by x". This means that there is a $\mathbb{Z}$-linear map $MU_*(\mathbb{C}P_+^\infty) \to K_*(MU)$ sending x^n to u^{n-1}. We deduce that x^n is not divisible (by integers) in $MU_*(\mathbb{C}P_+^\infty)$, whence the ring of primitive elements in $MU_*(\mathbb{C}P_+^\infty)$ is precisely $\mathbb{Z}[x]$. This is a simplified proof of the result which was first proved by D. Segal in [17], Theorem 2.1.

Example 4.7. The Hurewicz homomorphism $h\colon MU_* \to K_*(MU)$ can be easily determined using the following commutative diagram:

$$\begin{tikzcd} MU_* \arrow[r, "\eta_R"] \arrow[dr, "h"'] & MU_*(MU) \arrow[d, "g_*"] \\ & K_*(MU) \end{tikzcd} .$$

Hence, we can specify h as follows:

$$h: \qquad \zeta^{\phi} \mapsto \zeta^{\kappa} * \zeta^{\phi^K} \qquad \text{or} \qquad \mu^{\phi} \mapsto \mu^{\phi^K} * \mu^{\kappa}\,,$$

which means that

$$h(\phi_n) = \sum_{k=0}^{n} S(n+1,k+1)\phi_k^K u^{n-k}\,, \tag{4.8}$$

and

$$h(\overline{\phi}_n) = \sum_{k=0}^{n} (-1)^k k!\, s^{\phi^K}(n+1,k+1)\, u^k\,. \tag{4.9}$$

Example 4.10. We would now like to give a more explicit expression (than the usual one) for the coaction $\delta^K\colon K_*(MU) \to K_*(K) \otimes_{K_*} K_*(MU)$. Let us present first a simple way of computing the image of b_n^{MU} under the map $(g \wedge g)_*\colon MU_*(MU) \to K_*(K)$. Since g is a map of ring spectra, we have the commutative diagram

$$\begin{tikzcd} MU_* \arrow[r, "\eta_R"] \arrow[d, "g_*"'] & MU_*(MU) \arrow[d, "(g\wedge g)_*"] \\ K_* \arrow[r, "\eta_R^K"] & K_*(K) \end{tikzcd} .$$

Hence $(g\wedge g)_*(\phi_n^R) = v^n$. On the other hand, from (4.1) we obtain $\zeta^{\phi^{MU}} = \mu^{\phi} * \zeta^{\phi^R}$. By applying $(g \wedge g)_*$, we deduce

$$(g\wedge g)_* : \qquad \zeta^{\phi^{MU}} \mapsto \mu^{\kappa} * \zeta^{\kappa^R},$$

which means that

$$(g\wedge g)_*(b_n^{MU}) = \frac{B_n^{\kappa}(\kappa^R)}{(n+1)!} = \frac{(v-u)\ldots(v-nu)}{(n+1)!}\,.$$

Let us now recall the commutative diagram

$$\begin{tikzcd}
MU_*(MU) \arrow[r, "\delta"] \arrow[d, "g_*"'] & MU_*(MU) \otimes_{MU_*} MU_*(MU) \arrow[d, "(g\wedge g)_*\otimes g_*"] \\
K_*(MU) \arrow[r, "\delta^K"] & K_*(K) \otimes_{K_*} K_*(MU)
\end{tikzcd} .$$

Hence, we have

$$\delta^K : \qquad \zeta^{\phi^K} \mapsto \mu^{\kappa} * \zeta^{\kappa^R} \circledast \zeta^{\phi^K},$$

which means that

$$\begin{aligned}
\delta^K(b_n^K) &= \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{\pi \leq \sigma \in \Pi_{n+1,k}} \mu^{\kappa}(\widehat{0}, \pi)\, \zeta^{\kappa^R}(\pi, \sigma) \right) \otimes b_{k-1}^K \\
&= \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{i=k}^{n+1} \sum_{\pi \in \Pi_{n+1,i}} \mu^{\kappa}(\widehat{0}, \pi) \left(\sum_{\pi \leq \sigma \in \Pi_{n+1,k}} \zeta^{\kappa^R}(\pi, \sigma) \right) \right) \otimes b_{k-1}^K .
\end{aligned}$$

In consequence, we have the following result:

$$\delta^K(b_n^K) = \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} \left(\sum_{i=k}^{n+1} s(n+1, i)\, S(i, k)\, u^{n+1-i}\, v^{i-k} \right) \otimes b_{k-1}^K . \tag{4.11}$$

5. The Hurewicz Homomorphism $MU_* \to K_*(MU)$

In this section, we intend to compute the images of the coefficients $f_{n,m}^b$ of the universal formal group law under the Hurewicz homorphism h mentioned in the title; these coefficients are related to the coefficients $F_{n,m}^{\phi}$ of the universal Hurwitz group law by $F_{n,m}^{\phi} = n!m!f_{n,m}^b$ (see [8]). An algorithm for this computation appears in [1], but no closed formula is given. All the set partitions considered in this section lie in the lattice Π_{n+m}, with meet denoted by $\wedge$. According to 3.1, we have

$$F_{n,m}^{\phi} = \sum_{\sigma \leq \sigma_0} \mu^{\phi}(\widehat{0}, \sigma)\, \zeta^{\phi}(\sigma, \widehat{1}),$$

where σ_0 is the partition $\{[n], n+[m]\}$ of $[n+m]$. Using (4.8) and (4.9), we have

$$\begin{aligned}
h(F_{n,m}^{\phi}) &= \sum_{\pi \leq \sigma \leq \tau,\ \sigma \leq \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi)\, \mu^{\phi^K}(\pi, \sigma)\, \zeta^{\kappa}(\sigma, \tau)\, \zeta^{\phi^K}(\tau, \widehat{1}) \\
&= \sum_{\pi \leq \sigma \leq \sigma_0 \wedge \tau} \mu^{\phi^K}(\widehat{0}, \pi)\, \mu^{\phi^K}(\pi, \sigma)\, \zeta^{\kappa}(\sigma, \sigma_0 \wedge \tau)\, \zeta^{\kappa}(\sigma_0 \wedge \tau, \tau)\, \zeta^{\phi^K}(\tau, \widehat{1}) .
\end{aligned}$$

If we sum only over $\pi < \sigma_0 \wedge \tau$ we get 0, since we have the factor $(\mu^{\phi^K} * \zeta^\kappa)(\pi, \sigma_0 \wedge \tau) = 0$. Hence

$$\begin{aligned} h(F^\phi_{n,m}) &= \sum_\tau \mu^{\phi^K}(\widehat{0}, \sigma_0 \wedge \tau)\, \zeta^\kappa(\sigma_0 \wedge \tau, \tau)\, \zeta^{\phi^K}(\tau, \widehat{1}) \\ &= \sum_{\pi \le \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi) \left(\sum_{\tau\,:\,\sigma_0 \wedge \tau = \pi} \zeta^{\phi^K}(\tau, \widehat{1})\, u^{|\pi| - |\tau|} \right) \\ &= \sum_{\pi \le \sigma_0} \mu^{\phi^K}(\widehat{0}, \pi) \left(\sum_{k=0}^{\min\{n_\pi, m_\pi\}} k! \binom{n_\pi}{k} \binom{m_\pi}{k} \phi^K_{|\pi|-k-1} u^k \right) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} s^{\phi^K}(n,i)\, s^{\phi^K}(m,j) \left(\sum_{k=0}^{\min\{i,j\}} k! \binom{i}{k} \binom{j}{k} \phi^K_{i+j-k-1}\, u^k \right) , \end{aligned}$$

where n_π and m_π denote the number of blocks of the partition $\pi \le \sigma_0$ contained in $[n]$ and $n + [m]$, respectively; the third equality follows by counting the partitions τ with $\sigma_0 \wedge \tau = \pi$: concentrating on such partitions with precisely k blocks intersecting both $[n]$ and $n + [m]$, there are $\binom{n_\pi}{k}\binom{m_\pi}{k}$ ways of choosing the blocks of π to be amalgamated, and $k!$ ways of matching them. Finally, we have the following result:

Proposition 5.1. *The images of the coefficients of the universal formal group law $f^b_{n,m}$ under the Hurewicz homomorphism h are specified by*

$$h(f^b_{n,m}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta^{b^K}_{n,i}\, \beta^{b^K}_{m,j} \left(\sum_{k=0}^{\min\{i,j\}} \binom{i+j-k}{k,\; i-k,\; j-k} b^K_{i+j-k-1}\, u^k \right) .$$

6. Congruences in MU_*

In this section, we prove some congruences for the Φ_*-Stirling numbers $S^\phi(n,k)$ in $MU_{2(n-k)}$ modulo a prime p. The main tool will be the Hattori-Stong theorem and the periodicity modulo p of the classical Stirling numbers of the second kind, for which there is a nice proof using group actions (see [14]):

$$S(n,k) \equiv S(n-p+1,k) \pmod{p}, \quad \text{for } n > p-1 . \tag{6.1}$$

The Hattori-Stong theorem essentially says that the Hurewicz homomorphism $h\colon MU_* \to K_*(MU)$ is integrality preserving, i.e. for all $\alpha \in MU_* \otimes \mathbb{Q}$ we have that $\alpha \in MU_*$ if and only if $(h \otimes 1)(\alpha) \in K_*(MU)$. This turns out to be a purely algebraic statement, and such a proof can be found in [3].

Proposition 6.2. *We have the following congruences in MU_*:*

$$S^\phi(n, p-1) \equiv \begin{cases} 0 \pmod{p} & \text{if } n \not\equiv 0 \pmod{p-1} \\ \phi_{n-p+1} \pmod{p} & \text{otherwise,} \end{cases} \tag{6.3}$$

in $MU_{2(n-p+1)}$*;*

$$S^\phi(n,p-2) \equiv \begin{cases} 0 \pmod p & \text{if } n \not\equiv 0 \pmod{p-1} \\ \phi_{n-p+2} \pmod p & \text{otherwise,} \end{cases} \tag{6.4}$$

in $MU_{2(n-p+2)}$*, if* $p \geq 3$*;*

$$S^\phi(n,p-3) \equiv \begin{cases} 0 \pmod p & \text{if } n \not\equiv 0,-1,-2 \pmod{p-1} \\ \phi_{n-p+3} \pmod p & \text{if } n \equiv -2 \pmod{p-1} \\ 3\phi_{n-p+3} \pmod p & \text{if } n \equiv -1 \pmod{p-1} \\ 3\phi_1\phi_{n-p+2} - \phi_{n-p+3} \pmod p & \text{if } n \equiv 0 \pmod{p-1}, \end{cases} \tag{6.5}$$

in $MU_{2(n-p+3)}$*, if* $p \geq 5$.

Proof. Let us compute the image of $S^\phi(n,k)$ under the Hurewicz homomorphism $h\colon MU_* \to K_*(MU)$. By (4.8) we have

$$\begin{aligned} h(S^\phi(n,k)) &= \sum_{\pi \leq \sigma \in \Pi_{n,k}} \zeta^\kappa(\widehat{0},\pi)\,\zeta^{\phi^K}(\pi,\sigma) \\ &= \sum_{i=k}^{n} \sum_{\pi \in \Pi_{n,i}} \zeta^{\phi^K}(\widehat{0},\pi) \left(\sum_{\pi \leq \sigma \in \Pi_{n,k}} \zeta^{\phi^K}(\pi,\sigma) \right) \\ &= \sum_{i=k}^{n} S(n,i) S^{\phi^K}(i,k) u^{n-k}. \end{aligned}$$

We will first show that if $i \geq p$ and $k \leq p-1$, then $S^{\phi^K}(i,k) \equiv 0 \pmod p$. Consider a partition $\sigma \in \Pi_{i,k}$ with $k \leq p-1$ blocks and type $\tau^{\phi^K}(\sigma) = (\phi_1^K)^{\lambda_2} \ldots (\phi_{j-1}^K)^{\lambda_j}$. If $j \geq p$, then $\tau^{\phi^K}(\sigma)$ is divisible by p in $K_*(MU)$ since ϕ_{p-1}^K is. If $j < p$, then there are

$$\frac{i!}{(2!)^{\lambda_2} \ldots (j!)^{\lambda_j} (k-\lambda_2-\ldots-\lambda_j)!\lambda_2! \ldots \lambda_j!}$$

partitions in $\Pi_{i,k}$ having the same type as σ; but this number is divisible by p under the above hypothesis.

Now consider the image of ϕ_m under h given by (4.8). Using (6.1), we deduce that

$$h(\phi_m) \equiv \begin{cases} u^m \pmod p & \text{if } m \equiv 0 \pmod{p-1} \\ u^m + \phi_1^K u^{m-1} \pmod p & \text{if } m \equiv 1 \pmod{p-1} \\ u^m + 3\phi_1^K u^{m-1} + \phi_2^K u^{m-2} \pmod p & \text{if } m \equiv 2 \pmod{p-1}, \end{cases} \tag{6.6}$$

in $K_{2m}(MU)$.

According to the above remarks, and using (6.1) again, we have the following congruences, implying (6.3), (6.4), and (6.5), respectively:

$$h(S^\phi(n,p-1)) \equiv S(n,p-1)u^{n-p+1} \equiv \begin{cases} 0 \pmod p & \text{if } n \not\equiv 0 \pmod{p-1} \\ u^{n-p+1} \pmod p & \text{otherwise;} \end{cases}$$

$$h(S^{\phi}(n,p-2)) \equiv S(n,p-2)u^{n-p+2} + S(n,p-1)S(p-1,p-2)\phi_1^K u^{n-p+1}$$
$$\equiv \begin{cases} 0 \pmod p & \text{if } n \not\equiv 0,-1 \pmod{p-1} \\ u^{n-p+2} \pmod p & \text{if } n \equiv -1 \pmod{p-1} \\ u^{n-p+2} + \phi_1^K u^{n-p+1} \pmod p & \text{if } n \equiv 0 \pmod{p-1}; \end{cases}$$

$$h(S^{\phi}(n,p-3)) \equiv S(n,p-3)u^{n-p+3} + S(n,p-2)S(p-2,p-3)\phi_1^K u^{n-p+2}$$
$$+S(n,p-1)S^{\phi^K}(p-1,p-3)u^{n-p+1}$$
$$\equiv \begin{cases} 0 \pmod p & \text{if } n \not\equiv 0,-1,-2 \pmod{p-1} \\ u^{n-p+3} \pmod p & \text{if } n \equiv -2 \pmod{p-1} \\ 3u^{n-p+3} + 3\phi_1^K u^{n-p+2} \pmod p & \text{if } n \equiv -1 \pmod{p-1} \\ 2u^{n-p+3} + 3\phi_1^K u^{n-p+2} + (3(\phi_1^K)^2 - \phi_2^K)u^{n-p+1} \pmod p & \text{if } n \equiv 0 \pmod{p-1}. \end{cases}$$

We have used the following facts:

$$S(p-1,p-2) = \binom{p-1}{2} \equiv 1 \pmod p, \qquad S(p-2,p-3) = \binom{p-2}{2} \equiv 3 \pmod p,$$

$$S^{\phi^K}(p-1,p-3) = \frac{1}{2}\binom{p-1}{2}\binom{p-3}{2}(\phi_1^K)^2 + \binom{p-1}{3}\phi_2^K \equiv 3(\phi_1^K)^2 - \phi_2^K \pmod p$$

in $K_4(MU)$. □

Congruence (6.3) was proved in [11] using arguments related to the universal formal group law; it had an essential role therein for proving the universal von Staudt theorems. The advantage of this proof is that it also provides (6.4) and (6.5), which seem to be new.

7. Combinatorial Models for the Dual of the Steenrod Algebra

In this section, we discuss some connections between the Hopf algebras Φ_* and $(H\cdot\Phi)_*$, and the dual of the Steenrod algebra. Some of the results are known; nevertheless, we believe that the combinatorial proofs presented here provide new insights.

Consider a prime p, and let $\phi_{(n)} := \phi_{p^n-1}$. In [13] it is shown that the subalgebra $\mathbb{Z}/p[\phi_{(1)}, \phi_{(2)}, \ldots]$ of the mod p Faà di Bruno Hopf algebra $\Phi_* \otimes \mathbb{Z}/p$ is actually a sub-Hopf algebra, isomorphic to the polynomial part of the dual of the mod p Steenrod algebra; the proof is based on number-theoretical arguments. We present here an alternative proof, which is purely combinatorial and was inspired from [14].

Consider the cyclic group C_{p^n} acting on $[p^n]$, and hence on the partition lattice Π_n. We want to determine the partitions fixed by every element of C_{p^n}. If σ is such a partition, then C_{p^n} acts on its blocks. Let g be the cycle $(1,2,\ldots,p^n)$, let B be the block of σ containing 1, and let $\langle g^{p^k}\rangle$ be the stabilizer of B. From $g^{p^k}B = B$, we deduce that $\{1, 1+p^k, \ldots, 1+(p^{n-k}-1)p^k\} \subseteq B$. Furthermore, the sets $\{i, i+p^k, \ldots, i+(p^{n-k}-1)p^k\}$, for $i = 1, 2, \ldots, p^k$, all lie in different blocks of σ, whence they are precisely the blocks of σ. In consequence, we have

$n+1$ partitions fixed by each element of C_{p^n}, one for each $k=0,1,\dots,n$. The orbit of each partition which is not of the above type has p^i elements, where $i>0$. Finally, since all the partitions in the same orbit have the same type, (2.1) becomes

$$\delta(\phi_{(n)})=\sum_{k=0}^{n}\phi_{(n-k)}^{p^k}\otimes\phi_{(k)}\qquad\text{in }\mathbb{Z}/p[\phi_{(1)},\phi_{(2)},\dots]\,. \tag{7.1}$$

Let us now consider the Hopf algebra $P_*:=(H\cdot\Phi)_*\otimes\mathbb{Z}/p$, and the ideal J_* generated by the elements b_i, $i\neq p^n-1$. Writing $b_{(n)}$ for b_{p^n-1}, we clearly have an isomorphism of algebras $P_*/J_*\cong\mathbb{Z}/p[b_{(1)},b_{(2)},\dots]$. We intend to show that J_* is also a coideal (and hence a Hopf ideal), and that in the Hopf algebra $\mathbb{Z}/p[b_{(1)},b_{(2)},\dots]$ we have

$$\delta(b_{(n)})=\sum_{k=0}^{n}b_{(n-k)}^{p^k}\otimes b_{(k)}\,. \tag{7.2}$$

We will prove these statements together. Consider arbitrary integers $m,k\geq 1$ such that $m\geq p^k$. A partition in $I\Pi_{m,p^k}$ can be represented by a p^k-tuple $(\lambda_1,\dots,\lambda_{p^k})$ with $\lambda_i\geq 1$ and $\sum_{i=1}^{p^k}\lambda_i=m$. The cyclic group C_{p^k} acts on $I\Pi_{m,p^k}$ in the obvious way, and there is at most one partition fixed by each element of C_{p^k}, namely the one with equal block sizes, if p^k divides m; the sizes of all the other orbits are non-zero powers of p. This shows that if k_0 is highest power of p dividing m, then

$$\delta(b_{m-1})\;\in\;P_*\otimes J_*+\sum_{k=0}^{k_0}b_{m/p^k-1}^{p^k}\otimes b_{(k)}\,,$$

whence the desired statements follow.

In consequence, we have proved:

Proposition 7.3. *The polynomial part of the dual of the mod p Steenrod algebra is isomorphic to the sub-Hopf algebra $\mathbb{Z}/p[\phi_{(1)},\phi_{(2)},\dots]$ of the mod p Faà di Bruno Hopf algebra $\Phi_*\otimes\mathbb{Z}/p$, as well as to the quotient of $(H\cdot\Phi)_*\otimes\mathbb{Z}/p$ (i.e. the dual of the Landweber-Novikov algebra tensored with $\mathbb{Z}/p$) by the Hopf ideal J_*.*

Let R_* be an evenly graded commutative ring of characteristic p. Since P_*/J_* is a Hopf algebra, then the set $\mathrm{Hom}(P_*/J_*,R_*)$ is a group under convolution. We can now derive from Theorem 2.3 (2) an analogue for *p-typical power series* mod p.

Proposition 7.4. *The set of formal power series in $R^*[[X]]$ of the form*

$$r(X)=\sum_{i\geq 1}r_{p^i-1}X^{p^i}$$

(i.e. p-typical power series) form a group under substitution. There is an anti-isomor-phism from $\mathrm{Hom}(\mathbb{Z}/p[b_{(1)}, b_{(2)}, \dots], R_*)$ *to this group, specified by* $\zeta^r \mapsto r(X)$.

In consequence, Lagrange inversion for p-typical power series in $R^*[[X]]$ is equivalent to computing the antipode of $\mathrm{Hom}(\mathbb{Z}/p[b_{(1)}, b_{(2)}, \dots], R_*)$, which is much easier than computing the antipode of $\mathrm{Hom}((H \cdot \Phi)_*, R_*)$.

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