DUAL HOPF ORDERS IN GROUP RINGS OF ELEMENTARY ABELIAN $p$-GROUPS

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Let $K$ be a finite extension of $\mathbb{Q}_p$ with valuation ring $R$. In this paper we construct Hopf orders over $R$ in $KG$ where $G$ is an elementary abelian $p$-group of order $p^n$. Complete classifications of such orders are known only for $n = 1, 2$. The only previous work on orders in $KG$ for arbitrary $n$ are Raynaud’s construction in 1974 [Ra74] and the constructions in [CS98] and [CG98].

In Section 1, we apply the strategy of utilizing Larson orders inside arbitrary Hopf orders, used in [GC98], to construct new classes of triangular Hopf orders in $KG$ for arbitrary $n$. In Section 2, we extend the construction to obtain pairs of dual triangular Hopf orders in $KG$. Here, ”triangular” is in the sense of [UC04]—the unit parameters are conveniently described as entries of a lower triangular matrix. In Section 3 we construct dual pairs of triangular Hopf orders using the truncated exponential function introduced in [CG98]. To do the construction requires subjecting the unit parameters to an additional set of inequalities beyond those required in Section 2. The paper concludes with examples, computed using the simplex algorithm, that show that not every pair of dual Hopf orders constructed in Section 2 can be obtained via the truncated exponential construction of Section 3.

1. Hopf orders

Let $K$, $R$ be as noted above. Let $\pi$ be a parameter for (generator of the maximal ideal of) $R$, let $e$ be the absolute ramification index of $K$, assume $K$ contains a primitive $p$th root of unity $\zeta$, and let $e' = \text{ord}(\zeta - 1) = e/(p-1)$. Let $G$ be an elementary abelian $p$-group of order $p^n$, $G = G_1 \times \ldots G_n$ with $G_r = \langle \sigma_r \rangle$ cyclic of order $p$. In $KG_r = K\langle \sigma_r \rangle$, let

$$e_j^{(r)} = \frac{1}{p} \sum_{i=1}^{p-1} \zeta^{-ij} \sigma_r^i,$$
$j = 1, \ldots, p - 1$, be the primitive idempotents. For $u$ in $K$ let

$$a_u^{(r)} = \sum_{k=0}^{p-1} u^k e_k^{(r)}.$$ 

Then $a_u^{(r)}$ is a multiplicative homomorphism from $K^\times$ to $KG_r^{\times}$ satisfying $a_u^{(r)} = \sigma_r$.

Let $i_1, \ldots, i_n$ be valuation parameters: numbers satisfying $0 \leq i_j \leq e'$ for all $j$, and denote $i'_j = e' - i_j$ as usual. Assume $i_r \leq pi_j$ for all $r$ and all $j < r$, and also that $i'_s \leq pi'_k$ for all $s$ and all $k > s$. Let $U = (u_{i,j})$ be a lower triangular matrix with diagonal entries $= \zeta$. Let $a_{i,j} = a_{u_{i,j}}^{(j)}$ in $KG_j$.

Then $a_{i,j}^{(j)} = a_{u_{i,j}}^{(j)} = \sigma_j$. Consider the $R$-algebra

$$H_n = R[\frac{\sigma_1 - 1}{\pi^{i_1}}, \frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}}, \ldots, \frac{a_{n,1}a_{n,2} \cdots a_{n,n-1}\sigma_n - 1}{\pi^{i_n}}].$$

In this section we find conditions on the entries of $U$ for $H_n$ to be a Hopf order over $R$ in $KG$. Our approach uses the following result of Greither [Gr92], for which a convenient reference is [C00, (31.3)]:

**Proposition 1.** Let $0 \leq i, k \leq e'$ and $k < pi$. Let $G, G'$ be cyclic of order $p$ with generators $\sigma, \sigma'$, respectively. For $v$ in $R$, let $a_v$ be in $KG$ and $t = \frac{a_v^{\sigma' - 1}}{\pi^k}$. Let $H(i) = R[\frac{\sigma - 1}{\pi^i}]$ in $KG$. Then $E_v = H(i)[t]$ is a Hopf order in $K[G \times G']$, free of rank $p$ as an $H(i)$-module with power basis $1, t, \ldots, t^{p-1}$, if and only if

$$\text{ord}(v - 1) \geq i' + \frac{k}{p}$$

and

$$\text{ord}(v - 1) \geq i' + k.$$ 

We proceed to construct the order $H_n$ in $KG, G = G_1 \times \ldots \times G_n$, inductively. Let

$$H_1 = R[\frac{\sigma_1 - 1}{\pi^{i_1}}] = R[\frac{a_{1,1} - 1}{\pi^{i_1}}],$$

a Larson order in $KG_1$, and set $\mu_1 = i_1$, the first Larson parameter. Recalling the matrix $U$, let

$$H_2 = H_1[\frac{a_{2,1}\sigma_2 - 1}{\pi^{i_2}}] = H_1[\frac{a_{2,1}a_{2,2} - 1}{\pi^{i_2}}].$$

By Proposition 1, this is a Hopf order in $K[G_1 \times G_2]$ iff

$$\text{ord}(u_{2,1} - 1) \geq \frac{i'_1}{p} + i_2.$$
and 
\[ \text{ord}(u_{2,1} - 1) \geq i'_1 + \frac{i_2}{p}. \]

Given that \( H_2 \) is a Hopf order, we have 
\[ H_2 \cap K[\sigma_1] = H_1 = H(i_1) = R[\frac{\sigma_1 - 1}{\pi^{i_1}}], \]
and we need to find some \( \mu_2 \) so that 
\[ H(\mu_2) = R[\frac{\sigma_2 - 1}{\pi^{\mu_2}}] \subseteq H_2 \cap K[\sigma_2]. \]

**Proposition 2.** \( H(\mu_2) \subseteq H_2 \cap K[\sigma_2] \) if \( \mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\} \).

**Proof.** We have 
\[ \frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} = \frac{a_{2,1}(\sigma_2 - 1)}{\pi^{\mu_2}} + \frac{a_{2,1} - 1}{\pi^{\mu_2}}. \]
Assume that \( \mu_2 \leq i_2 \), so that \( \frac{a_{2,1}\sigma_2 - 1}{\pi^{\mu_2}} \) is in \( H_2 \). Then \( a_{2,1} \) is a unit of \( H_2 \), and so \( \frac{a_{2,1} - 1}{\pi^{\mu_2}} \) will be in \( H_2 \) if and only if \( \frac{a_{2,1} - 1}{\pi^{\mu_2}} \) is in \( H_1 \). But by [UC, Proposition 2.1],
\[ R[\frac{a_{2,1} - 1}{\pi^{\mu_2}}] \subset H(i_1) \]
iff
\[ \text{ord}(u_{2,1} - 1) - \mu_2 \geq \varepsilon' - i_1 = i'_1, \]
or \( \mu_2 \leq \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\} \).

We call \( \mu_2 \) the second Larson parameter. (One can show that for \( \mu = \min\{i_2, \text{ord}(u_{2,1} - 1) - i'_1\} \), then \( H(\mu_2) \) is the largest Larson order in \( H_2 \cap K[\sigma_2] \).

We proceed by induction. Suppose we have found conditions on the entries of the first \( r - 1 \) rows of the matrix \( U \) so that \( H_{r-1} \) is an \( R \)-Hopf order in \( K[G_1 \times \ldots \times G_{r-1}] \) in such a way that for \( 2 \leq j \leq r - 1 \), \( H_j \) is free of rank \( p \) over \( H_{j-1} \) on powers of the algebra generator \( t_j = \frac{a_{j,1}a_{j,2} \ldots a_{j,i_{r-1},j,i_{r-1},i_{r-1},j,i_{r-1}} - 1}{\pi^{i_j}} \) of \( H_j \) over \( H_{j-1} \). Suppose also we have found \( r - 1 \) Larson parameters \( \mu_1 = i_1, \mu_2, \ldots, \mu_{r-1} \) so that 
\[ H_j \cap K[G_j] = H_{r-1} \cap K[G_j] \supseteq H(\mu_j) = R[\frac{\sigma_j - 1}{\pi^{\mu_j}}]. \]
Consider 
\[ H_r = H_{r-1}[\frac{a_{r,1}a_{r,2} \ldots a_{r,r-1} - 1}{\pi^{i_r}}] = H_{r-1}[\frac{a_{r,1}a_{r,2} \ldots a_{r,r-1}a_{r,r} - 1}{\pi^{i_r}}]. \]
For \( H_r \) to be free of rank \( p \) over \( H_{r-1} \) with power basis generated by
\[ t = \frac{a_{r,1}a_{r,2} \ldots a_{r,r-1} - 1}{\pi^{i_r}}. \]
it suffices that
\[
\frac{a_{r,1}^p a_{r,2}^p \cdots a_{r,r-1}^p - 1}{\pi^{\mu_i}} \in H_{r-1},
\]
which follows if
\[
\frac{a_{u,r,j}^p - 1}{\pi^{\mu_i}} \in H(\mu_j) \subset H_{r-1} \cap KG_j
\]
for all \( j < r \), which in turn follows from
\[
\text{ord}(u_{r,j}^p - 1) - pi_r \geq \varepsilon' - \mu_j
\]
that is,
\[
\text{ord}(u_{r,j} - 1) \geq \frac{\mu_j'}{p} + i_r
\]
for \( j = 1, \ldots, r-1 \).

To show that \( H_r \) is a Hopf order, that is, the comultiplication on \( KG \) maps \( H_r \) to \( H_r \otimes H_r \), we need
\[
\Delta(a_{r,1} \cdots a_{r,r-1}) \equiv a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} \quad (\text{mod } \pi^{i_r} H_{r-1} \otimes H_{r-1}).
\]
Now
\[
\Delta(a_{r,1} \cdots a_{r,r-1}) = \Delta(a_{r,1}) \cdots \Delta(a_{r,r-1})
\]
and
\[
a_{r,1} \cdots a_{r,r-1} \otimes a_{r,1} \cdots a_{r,r-1} = (a_{r,1} \otimes a_{r,1}) \cdots (a_{r,r-1} \otimes a_{r,r-1}).
\]
So it suffices that for each \( j \),
\[
\Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \quad (\text{mod } \pi^{i_r} H_{r-1} \otimes H_{r-1}).
\]
But \( a_{r,j} \in H_{r-1} \cap KG_j \supseteq H(\mu_j) \). So it suffices that
\[
\Delta(a_{r,j}) \equiv a_{r,j} \otimes a_{r,j} \quad (\text{mod } \pi^{i_r} H(\mu_j) \otimes H(\mu_j)),
\]
which holds if \( i_r \leq p\mu_j \) and
\[
\text{ord}(u_{r,j} - 1) \geq \mu_j' + \frac{i_r}{p}.
\]
Thus we need
\[
i_r \leq p\mu_j
\]
for all \( j = 1, \ldots, r-1 \), and
\[
\text{ord}(u_{r,j} - 1) \geq \mu_j' + \frac{i_r}{p}
\]
for \( j = 1, \ldots, r-1 \).

To complete the inductive construction, we need a parameter \( \mu_r \) so that
\[
H_r \cap K[G_r] \supseteq H(\mu_r).
\]
We have

\[
\frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{\mu_r}}
\]

\[
= \frac{a_{r,1} - 1}{\pi^{\mu_r}} + \sum_{i=2}^{r-1} (a_{r,1}a_{r,2} \cdots a_{r,i-1})(a_{r,i} - 1)\frac{\sigma_r - 1}{\pi^{\mu_r}} + (a_{r,1}a_{r,2} \cdots a_{r,r-1})(\sigma_r - 1)\frac{\sigma_r - 1}{\pi^{\mu_r}}.
\]

Now

\[
\frac{a_{r,1}a_{r,2} \cdots a_{r,r-1}\sigma_r - 1}{\pi^{\mu_r}}
\]

is in \( H_r \) if \( \mu_r \leq i_r \). If

\[
\frac{a_{r,k} - 1}{\pi^{\mu_r}} \in H_k
\]

for \( k = 1, \ldots, r - 1 \), then the \( a_{r,k} \) will be units of \( H_r \), and so

\[
\frac{\sigma_r - 1}{\pi^{\mu_r}} \in H_r,
\]

hence

\[
R\left[\frac{\sigma_r - 1}{\pi^{\mu_r}}\right] \subseteq H_r \cap K[G_r].
\]

Now since

\[
R\left[\frac{a_{r,k} - 1}{\pi^{\mu_r}}\right] = R\left[\frac{\sigma_k - 1}{\pi^{\nu_r}}\right]
\]

with \( \text{ord}(u_{r,k} - 1) - \mu_r = e' - \nu_k \) by [UC04, Proposition 2.1], and

\[
R\left[\frac{\sigma_k - 1}{\pi^{\nu_r}}\right] \subseteq H_k
\]

if \( \nu_k \leq \mu_k \), it follows that

\[
\frac{a_{r,k} - 1}{\pi^{\mu_r}} \in H_k
\]

if

\[
\text{ord}(u_{r,k} - 1) - \mu_r \geq e' - \mu_k
\]

for \( k = 1, \ldots, r - 1 \). Thus we call \( \mu_r \) an \( r \)th Larson parameter if \( \mu_r \) satisfies

\[
\mu_r \leq i_r
\]

and

\[
\text{ord}(u_{r,k} - 1) - \mu_r \geq e' - \mu_k
\]

for \( k = 1, \ldots, r - 1 \). (Note that we have not shown that if \( \mu_r \) is maximal satisfying those inequalities, then

\[
R\left[\frac{\sigma_r - 1}{\pi^{\mu_r}}\right] = H_r \cap K[G_r],
\]
and so we refer to $\mu_1, \ldots, \mu_r$ as Larson parameters, not the Larson parameters. Finding such a $\mu_r$ remains open for $r \geq 3$.)

To summarize:

**Theorem 3.** Given $i_1, \ldots, i_n$ with $0 \leq i_r \leq e'$ for all $r$ and $i_r \leq p\mu_s$ for $r > s$, suppose $U = (u_{r,s})$ is a lower triangular $n \times n$ matrix with entries in $R$ and diagonal entries $\zeta$. Set $H_0 = R$ and for $1 \leq r \leq n$, define $H_r$ by

$$H_r = H_{r-1} \llbracket t_r \rrbracket,$$

where

$$t_r = \frac{a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1}{\pi^r}.$$

Then $H_r$ is free over $H_{r-1}$ with basis $\{1, t_r, \ldots, t_{p-r-1} \}$ and $H(U) = H_n$ is a Hopf order with Larson parameters $\mu_1 = i_1, \mu_2, \ldots, \mu_n$ if the following inequalities hold for all $r, s$ with $1 \leq s < r \leq n$:

$$i_r \leq p\mu_s,$$

$$\text{ord}(u_{r,s} - 1) \geq \frac{\mu_s'}{p} + i_r,$$

$$\text{ord}(u_{r,s} - 1) \geq \mu_s' + \frac{i_r}{p},$$

$$\mu_r \leq i_r,$$

$$\mu_r \leq \text{ord}(u_{r,s} - 1) - \mu_s'.$$

Thus given $i_1, \ldots, i_n$ satisfying the conditions of Theorem 3, the matrix $U$ yields a Hopf order if the valuations $\text{ord}(u_{r,s} - 1)$ of the $n(n-1)/2$ off-diagonal entries of $U$ and the $n - 1$ Larson parameters satisfy a collection of $2n^2 - 2n$ linear inequalities.

2. Duality

Here is the corresponding result for a potential dual:

**Theorem 4.** Given $i'_1, \ldots, i'_n$ with $0 \leq i'_s \leq e'$ for all $s$ and $i'_s \leq p\mu'_r$ for $s < r$, suppose $W = (w_{s,r})$ is an upper triangular $n \times n$ matrix with entries in $R$ and diagonal entries $\zeta$. Set $J_{n+1} = R$ and for $1 \leq s \leq n$, define $J_s$ by

$$J_s = H_{s+1} \llbracket q_s \rrbracket,$$

where

$$q_s = \frac{b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1}{\pi^{i_s}}.$$

Then $J_s$ is free over $J_{s+1}$ with basis $\{1, q_s, \ldots, q_{s-1}^{p-1} \}$ and $J(W) = J_1$ is a Hopf order with Larson parameters $\delta'_n = i_n, \delta'_{n-1}, \ldots, \delta'_1$ if the following
inequalities hold for all \( s, r \) with \( 1 \leq r < s \leq n \):

\[
i'_s \leq p\delta'_r
\]

\[
\text{ord}(w_{s,r} - 1) \geq \frac{\delta_r}{p} + i'_s
\]

\[
\text{ord}(w_{s,r} - 1) \geq \delta_r + \frac{i'_s}{p}
\]

\[
\delta'_s \leq i_s
\]

\[
\delta'_s \leq \text{ord}(w_{s,r} - 1) - \delta_r
\]

We now find conditions on \( U \) and \( W \) so that \( H(U) \) and \( J(W) \) are dual Hopf orders.

Let \( G = \langle \sigma_1 \rangle \times \ldots \times \langle \sigma_n \rangle \) and \( \hat{G} = \langle \gamma_1 \rangle \times \ldots \times \langle \gamma_n \rangle \) with

\[
\langle \sigma_r, \gamma_s \rangle = 1 \text{ if } s \neq r, \quad \zeta \text{ if } s = r.
\]

Then \( KG \) and \( K\hat{G} \) are dual group rings.

For \( x \) and \( y \) units in \( R \) the quantity

\[
G(x, y) = \frac{1}{p} \sum_{0 \leq i,j \leq p-1} \zeta_1^{-ij}x^iy^j,
\]

is the Gauss sum of \( x \) and \( y \) ([GC98]). Note that \( G(x,1) = 1 \). Also,

\[
G(\zeta_1^k, w) = \frac{1}{p} \sum_{i,j=0}^{p-1} \zeta_1^{-ij}w^j = \frac{1}{p} \sum_{j=0}^{p-1} \left( \sum_{i=0}^{(k-j)i} \zeta_1^{-i} \right) w^j = w^k.
\]

The Gauss sum arises in connection with duality because

\[
G(x, y) = \langle a_x, a_y \rangle
\]

(where \( a_x \in K\sigma_p, a_y \in K\hat{\sigma}_p \)), as is easily verified (cf. [GC98]).

Assuming that \( H(U) \), \( J(W) \) are Hopf orders in \( KG \) and \( K\hat{G} \), respectively, then by the choice of denominators (valuation parameters), \( J(W) \) will be the dual of \( H(U) \) iff \( \langle H(U), J(W) \rangle \subset R \), by a routine discriminant argument. Since both are Hopf orders, it suffices that the duality map applied to generators maps into \( R \), that is, for all \( r, s \),

\[
\left( \frac{a_{r,1}a_{r,2}\cdots a_{r,r-1}\sigma_r - 1}{\pi^i_r}, \frac{b_{s,n}b_{s,n-1}\cdots b_{s,s+1}\gamma_s - 1}{\pi^i_s} \right) \subset R,
\]

that is,

\[
D_{r,s} - 1 := \left( \frac{a_{r,1}a_{r,2}\cdots a_{r,r-1}\sigma_r}{\pi^i_r}, \frac{b_{s,n}b_{s,n-1}\cdots b_{s,s+1}\gamma_s}{\pi^i_s} - 1 \right) \in \pi^{i_r+i_s}R.
\]
One sees easily that $D_{r,s} = 1$ if $r < s$ and $D(r,r) = \zeta$, and so $D_{r,s} - 1$ is in $\pi^{i_r+i'_r}R$ if $r \leq s$. For $r > s$,
\[
D_{r,s} = \langle a_{r,s}, \gamma_s \rangle \langle a_{r,s+1}, b_{s,s+1} \rangle \cdots \langle a_{r,r-1}, b_{s,r-1} \rangle \langle a_{r,s}, \gamma_s \rangle
= u_{r,s} G(u_{r,s+1}, w_{s,s+1}) \cdots G(u_{r,r-1}, w_{s,r-1}) w_{s,r}
\]
(c.f. [GC98, Lemma 2.1]).

In order to construct $W$ so that $D_{r,s} \equiv 1 \pmod{\pi^{i_r+i'_r}}$ for $r > s$, we make the assumptions on the entries of $U$:

$$\text{ord}(u_{r,s} - 1) > 0$$
for all $r > s$, and

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)$$
for all $r > k > s$. The first assumption follows from the inequalities of Theorem 3 provided that $i_r > 0$. The second assumption implies that

$$\text{ord}(G(u_{r,k}, u_{k,s}) - 1) = \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e'$$
by [UC, Proposition 2.3].

Assuming these inequalities, we define the off-diagonal entries of $W$ inductively.

For all $r$ define $w_{r-1,r}$ by

$$D_{r,r-1} = u_{r,r-1} w_{r-1,r} = 1.$$ 
Then $D_{r,r-1} - 1 \in \pi^{i_r+i'_{r-1}}R$ and $\text{ord}(u_{r,r-1} - 1) = \text{ord}(w_{r-1,r} - 1)$.

Define $w_{r-2,r}$ by

$$D_{r,r-2} = u_{r,r-2} G(u_{r-1,r}, w_{r-2,r-1}) w_{r-2,r} = 1.$$ 
This definition makes sense because

$$\text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$
$$= \text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-1,r-2} - 1) - e'$$
$$> \text{ord}(u_{r,r-2} - 1) > 0$$
and so

$$\text{ord}(G(u_{r,r-1}, w_{r-2,r-1}) - 1) = \text{ord}(u_{r,r-1} - 1) + \text{ord}(w_{r-2,r-1} - 1) - e'$$
$$> \text{ord}(u_{r,r-2} - 1),$$
hence both $G(u_{r,r-1}, w_{r-2,r-1})$ and $u_{r,r-2}$ are units. Also,

$$\text{ord}(u_{r,r-2} - 1) = \text{ord}(w_{r-2,r} - 1)$$
by the isosceles triangle inequality applied to $0 = D_{r,r-2} - 1$ since

$$D_{r,r-2} - 1 = u_{r,r-2} G(u_{r,r-1}, w_{r-2,r-1}) (w_{r-2,r} - 1)$$
$$+ u_{r,r-2} (G(u_{r,r-1}, w_{r-2,r-1}) - 1) + (u_{r,r-2} - 1).$$
Assume that $w_{s,r}$ has been defined for all $r, s$ with $r - s = d > 0$ so that $\text{ord}(w_{s,r} - 1) = \text{ord}(u_{r,s} - 1)$. We have

$$u_{r+1,s}G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r})w_{s,r+1} = 1.$$ 

Now for all $k$ with $r + 1 > k > s$,

$$\text{ord}(u_{r+1,k} - 1) + \text{ord}(w_{s,k} - 1) - e' = \text{ord}(u_{r+1,k} - 1) - 1 + \text{ord}(u_{k,s} - 1) - e' > \text{ord}(u_{r+1,s} - 1) > 0,$$

hence for $r + 1 > k > s$, $G(u_{r+1,k}, w_{s,k})$ and $u_{r+1,s}$ are units of $R$. Therefore we may define $w_{s,r+1}$ by

$$D_{r+1,s} = u_{r+1,s}G(u_{r+1,s+1}, w_{s,s+1}) \cdots G(u_{r+1,r}, w_{s,r})w_{s,r+1} = 1.$$ 

Since

$$\text{ord}(G(u_{r+1,k}, w_{s,k}) - 1 = \text{ord}(u_{r+1,k} - 1) + \text{ord}(w_{s,k} - 1) - e' > \text{ord}(u_{r+1,s} - 1)$$

for all $k$ with $r + 1 > k > s$, it follows that

$$\text{ord}(u_{r+1,s} - 1) = \text{ord}(w_{s,r+1} - 1)$$

by the isosceles triangle inequality.

In this way we may define the entries of $W$ so that

$$D_{r,s} - 1 = 0$$

for all $r > s$, and so we obtain a dual pair of Hopf orders, $H(U)$ and $J(W)$, provided that both $H(U)$ and $J(W)$ are Hopf orders.

We collect the needed inequalities for both $H(U)$ and $J(W)$ to be Hopf orders and duals of each other.

**Theorem 5.** Let $i_1, \ldots, i_n$ be valuation parameters satisfying $0 \leq i_r \leq e'$ for all $r$ and $i_r \leq pi_s$ and $i'_r \leq pi'_s$ for all $r > s$. Let $U = (u_{r,s})$ be a lower triangular $n \times n$ matrix with entries in $R$ and diagonal entries $\zeta$. Define the upper triangular matrix $W = (w_{s,r})$ by $w_{s,s} = \zeta$ and for $r > s$,

$$u_{r,s}G(u_{r,s+1}, w_{s,s+1}) \cdots G(u_{r,r-1}, w_{s,r-1})w_{s,r} = 1.$$ 

Then $H(U)$ and $J(W)$ are a dual pair of Hopf orders with Larson parameters $\mu_1 = i_1, \mu_2, \ldots, \mu_n$, $\delta'_n = i_n, \delta'_n, \ldots, \delta'_1$ if the following
inequalities hold for all $1 \leq s < k < r \leq n$:

$$\text{ord}(u_{r,s} - 1) > 0$$

$$i_r \leq p\mu_j$$

$$\text{ord}(u_{r,s} - 1) \geq i_r + \frac{\mu'_s}{p}$$

$$\text{ord}(u_{r,s} - 1) \geq i_r + \frac{\mu'_s}{p}$$

$$\mu_r \leq i_r$$

$$\text{ord}(u_{r,s} - 1) \geq \mu_r + \mu'_s$$

$$i'_s \leq p\delta'_r$$

$$\text{ord}(u_{r,s} - 1) \geq \frac{\delta_r}{p} + i'_s$$

$$\text{ord}(u_{r,s} - 1) \geq \delta_r + \frac{i'_s}{p}$$

$$\delta'_s \leq i'_s$$

$$\text{ord}(u_{r,s} - 1) \geq \delta_r + \delta'_s$$

$$\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1).$$

When $i_1, \ldots, i_n$ satisfy the strict inequalities $0 < i_1, \ldots, i_n < e'$, then to obtain a dual pair of Hopf orders $H(U)$ and $J(W)$ in $KC^n_p$, $\hat{KC}^n_p$, resp., Theorem 5 requires that the $\binom{n}{2}$ variables \{ord$(u_{r,s} - 1)$|$r > s$\} and the $2(n-1)$ variables $\mu_2, \ldots, \mu_n, \delta_{n-1}, \ldots, \delta_1$ must satisfy a system of

$$2(2n^2 - 2n) + \binom{n}{3} = \frac{n^3 + 21n^2 - 22n}{6}$$

linear inequalities.

It is routine to obtain solutions to these inequalities. See Remark 11 below.

3. Truncated exponentials

Now we consider a variant construction of dual Hopf orders in $KC^n_p$, generalizing that of [GC98]. This construction uses the truncated exponential function,

$$\exp(x) = \sum_{i=0}^{p-1} x^i/i!,$$

which behaves well with respect to duality. (We never explicitly use the untruncated series and so will not require special notation.) Here is the first of two results that facilitate the use of the truncated exponential:
Lemma 6. For $x, y \in \pi^l R$ with $l \geq 1$, we have

$$\exp(x + y) \equiv \exp(x) \exp(y) \pmod{\pi^l R}$$

Proof. As noted in [GC, Remark 1.1], the proof is a matter of showing the difference of the two sides can be written as a power series of order at least $p$ with coefficients in $R$ in which all terms have valuation $\geq l$. □

Define $\lambda \in R$ by the equation $\exp(\lambda) = \zeta$. (This can be done explicitly by solving $\exp(x) = \zeta$ for $x$ modulo higher and higher powers of $\pi$.) From the definition of $\exp(-)$, we get that $\ord(\lambda) = \ord(\zeta - 1) = e'$. Then for $x$ in $K$, $\ord(\exp(\lambda x) - 1) = \ord(x) + e'$. Thus $\exp(\lambda x)$ is a unit of $R$ iff $\ord(\lambda x) \geq 1$, iff $\ord(x) \geq -e' + 1$.

Let $Y = (y_{r,s})$ be a lower triangular matrix of elements of $K$ with diagonal entries $y_{r,r} = 1$ and $\ord(y_{r,s}) > -e'$ for all $r > s$. Then the matrix $U = (u_{r,s})$ with $u_{r,s} = \exp(\lambda y_{r,s})$ for $r \geq s$, $u_{r,s} = 0$ for $r < s$, is lower triangular with entries in $R$ and diagonal entries $\zeta$. The valuation conditions on $U$ in order that $H(U)$ be a Hopf order as in Theorem 3 translate immediately to valuation conditions on $Y$, since $\ord(u_{r,s}) = \ord(y_{r,s}) + e'$.

Denote $H(U) = H^e(Y)$.

Similarly, let $Z = (z_{r,s})$ be an upper triangular matrix of elements of $K$ with diagonal entries $z_{r,r} = 1$ and $\ord(z_{s,r}) > -e'$ for all $s < r$. Then the matrix $W = (w_{s,r})$ with $w_{s,r} = \exp(\lambda z_{s,r})$ for $s \leq r$, $z_{s,r} = 0$ for $s > r$, is upper triangular with entries in $R$ and diagonal entries $\zeta$. The valuation conditions on $W$ in order that $J(W)$ be a Hopf order as in Theorem 4 translate immediately to valuation conditions on $Z$, since $\ord(w_{s,r}) = \ord(z_{s,r}) + e'$. Denote $J(W) = J^e(Y)$.

The attractiveness of using matrices $U$ and $W$ with entries that are truncated exponentials of entries in $Y, Z$, respectively, is that the dual of $H^e(Y)$ is $J^e(Z)$ where the transpose of $Z$ is the inverse of $Y$.

Along with Lemma 6 we need for duality the following extension of a result on Gauss sums ([GC98, Theorem 1.4]):

**Theorem 7.** Let $x, y$ be elements of $K$ with $\min\{\ord(\lambda x), \ord(\lambda y)\} = g$ where $0 < g \leq e'$. Then

$$G(\exp(\lambda x), \exp(\lambda y)) - \exp(\lambda xy) \in \pi^{(2p-1)g - (p-1)e'} R$$

**Proof.** Let $P(X, Y) = G(\exp(\lambda X), \exp(\lambda Y))$. Theorem 1.4 of [GC98] asserts that if $X, Y$ are indeterminates, then the polynomial $Q(X, Y) = P(X, Y) - \exp(\lambda XY)$ satisfies

$$Q(X, Y) = \pi^{pe'} F(X, Y)$$
where $F(X,Y)$ is a polynomial with coefficients in $R$. Now

\[
Q(X,Y) = \frac{1}{p} \sum_{i,j=0}^{p-1} \left( \sum_{m=0}^{p-1} \frac{(\lambda X)^m}{m!} \right)^i \zeta^{ij} \left( \sum_{n=0}^{p-1} \frac{(\lambda Y)^n}{n!} \right)^j - \sum_{k=0}^{p-1} \frac{(\lambda XY)^k}{k!}
\]

for some coefficients $s_{m,n} \in R$. Since $Q(X,Y)$ has coefficients in $\pi^{pe'} R$, it follows that for all $m + n = d \geq 0$,

\[
\text{ord}(s_{m,n}) - \text{ord}(p) + de' \geq pe',
\]

hence

\[
\text{ord}(s_{m,n}) \geq (2p - 1 - d)e'.
\]

Since $s_{m,n}$ is in $R$, we also have

\[
\text{ord}(s_{m,n}) \geq 0
\]

for all $m, n$.

Suppose $\min\{\text{ord}(\lambda x), \text{ord}(\lambda y)\} = g$ with $0 < g \leq e'$. Then for each $m, n$ with $m + n = d$,

\[
\text{ord}\left( \frac{1}{p} s_{m,n} (\lambda x)^m (\lambda y)^n \right) \geq -(p - 1)e' + gd + ((2p - 1) - d)e'
\]

for $d \leq 2p - 1$, and

\[
\text{ord}\left( \frac{1}{p} s_{m,n} (\lambda x)^m (\lambda y)^n \right) \geq -(p - 1)e' + gd
\]

for $d \geq 2p - 1$. Thus the term with minimal valuation in $Q(x,y)$ has valuation

\[
-(p - 1)e' + (2p - 1)g,
\]

completing the proof. \[\square\]

Now we repeat the construction in section 2.

Assuming that $H^e(Y)$, $J^e(Z)$ are Hopf orders in $KG$ and $K\hat{G}$, respectively, then by the choice of denominators (valuation parameters), $J^e(Z)$ will be the dual of $H^e(Y)$ iff $\langle H^e(Y), J^e(Z) \rangle \subset R$. Since both are Hopf orders, it suffices that the duality map applied to generators lands in $R$, that is, for all $r, s$,

\[
\langle a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r - 1, b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s - 1 \rangle \pi^{ir} \pi^{is} \subset R.
\]

As before, set

\[
D_{r,s} = \langle a_{r,1} a_{r,2} \cdots a_{r,r-1} \sigma_r, b_{s,n} b_{s,n-1} \cdots b_{s,s+1} \gamma_s \rangle.
\]
We require that
\[ D_{r,s} - 1 \in \pi^{i_r+i'_s} R. \]
One sees easily that \( D_{r,s} = 1 \) if \( r < s \), and = \( \zeta \) if \( r = s \). For \( r > s \),
\[
D_{r,s} = \langle a_{r,s} \gamma_s \rangle \langle a_{r,s+1} b_{s+1} \rangle \cdots \langle a_{r-r+1} b_{r-r} \rangle \langle a_{r,s} \rangle
\]
\[
= P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r-r+1}, z_{s,r}) P(y_{r,r}, z_{s,r})
\]
(c.f. [GC98, Lemma 2.1]), where \( P(x, y) = G(\exp(\lambda X), \exp(\lambda Y)) \) as in Lemma 7. Thus we want
\[
P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdots P(y_{r-r+1}, z_{s,r}) P(y_{r,r}, z_{s,r}) \in \pi^{i_r+i'_s} R
\]
for all \( r > s \).

As in the previous section, in order to construct \( Z \) so that \( D_{r,s} \equiv 1 \) (mod \( \pi^{i_r+i'_s} R \)) for \( r > s \), we assume that for all \( r > k > s \), we have
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}).
\]

If \( u_{r,s} = \exp(\lambda y_{r,s}) \), then this assumption is equivalent to the assumption
\[
\text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) > e' + \text{ord}(u_{r,s} - 1)
\]
made in the construction of Theorem 5.

**Proposition 8.** Suppose \( Z^t = Y^{-1} \), and assume that for all \( r > k > s \),
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}).
\]
Then for all \( r > s \), \( \text{ord}(z_{s,r}) = \text{ord}(y_{r,s}) \).

**Proof.** For \( r > s \) we have
\[
y_{r,s} + y_{r,s+1} z_{s,s+1} + \ldots + y_{r-r+1} z_{s,r} + z_{s,r} = 0.
\]
Thus \( \text{ord}(y_{r-r+1}) = \text{ord}(z_{s,r}) \) for all \( r \). Proceeding by induction, assume \( \text{ord}(z_{s,t}) = \text{ord}(y_{r,t}) \) for \( r - t < r - s \), then
\[
\text{ord}(y_{r,s}) < \text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) = \text{ord}(y_{r,k} z_{s,k})
\]
for \( k = s + 1, \ldots, r - 1 \), so by the isosceles triangle inequality we have
\[
\text{ord}(y_{r,s}) = \text{ord}(z_{s,r}).
\]

Assume the entries of \( Y \) satisfy the hypotheses of Proposition 8.

Then by Theorem 7, for all \( r > k > s \),
\[
P(y_{r,k}, z_{s,k}) \equiv \exp(\lambda y_{r,k} z_{s,k}) \pmod{\pi^{(2p-1)g_{r,k,s}-(p-1)e'}}
\]
where
\[
g_{r,k,s} = \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda z_{s,k})\}
\]
\[
= \min\{\text{ord}(\lambda y_{r,k}), \text{ord}(\lambda y_{k,s})\}
\]
by Proposition 8. So assume

\[(2p - 1)\text{ord}(\lambda y_{r,k}) - (p - 1)e' \geq i_r + i'_s\]

and

\[(2p - 1)\text{ord}(\lambda y_{k,s}) - (p - 1)e' \geq i_r + i'_s\]

for all \( r > k > s \). Then modulo \( \pi^{i_r+i'_s}R \)

\[D_{r,s} = P(y_{r,s}, z_{s,s}) P(y_{r,s+1}, z_{s,s+1}) \cdot \ldots \cdot P(y_{r,r-1}, z_{s,r-1}) P(y_{r,r}, z_{s,r})\]

\[\equiv \exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1} z_{s,s+1}) \cdot \ldots \cdot \exp(\lambda y_{r,r-1} z_{s,r-1}) \exp(\lambda z_{s,r}).\]

Now we apply Lemma 6: assume that for all \( r \geq k \geq s \),

\[\text{ord}(\lambda y_{r,k} z_{s,k}) \geq \frac{i_r + i'_s}{p}.\]

Then

\[\exp(\lambda y_{r,s}) \exp(\lambda y_{r,s+1} z_{s,s+1}) \cdot \ldots \cdot \exp(\lambda y_{r,r-1} z_{s,r-1}) \exp(\lambda z_{s,r})\]

\[\equiv \exp(\lambda(y_{r,s} + y_{r,s+1} z_{s,s+1} + \ldots + y_{r,r-1} z_{s,r-1} + z_{s,r}))\]

\[= \exp(0) = 1 \pmod{\pi^{i_r+i'_s}}\]

and so for all \( r > s \),

\[D_{r,s} - 1 \in \pi^{i_r+i'_s}R,\]

which implies that

\[\langle H^e(Y), J^e(Z) \rangle \subseteq R.\]

We have shown nearly all of

**Theorem 9.** Suppose \( Y = (y_{r,s}) \) is an \( n \times n \) lower triangular matrix with entries in \( \pi^{-e'+1}R \) and diagonal entries 1. Suppose for all \( r > k > s \),

\[\text{ord}(y_{r,k}) + \text{ord}(y_{k,s}) > \text{ord}(y_{r,s}).\]

Let \( Z^t = Y^{-1} \), let \( U = (u_{r,s}) \) be lower triangular with \( u_{r,s} = \exp(\lambda y_{r,s}) \) for \( r \geq s \), and let \( W = (w_{s,r}) \) be upper triangular with \( w_{s,r} = \exp(\lambda z_{s,r}) \) for \( r \geq s \). If \( \{u_{r,s}, \mu_r, \delta_s^t\} \) satisfy the inequalities of Theorem 5 together with the inequalities

\[(2p - 1)\text{ord}(\lambda y_{r,k}) \geq (p - 1)e' + i_r + i'_s\]

\[(2p - 1)\text{ord}(\lambda y_{k,s}) \geq (p - 1)e' + i_r + i'_s\]

for all \( r > k > s \), then \( J(W) \) and \( H(U) \) are a dual pair of Hopf orders.

**Proof.** Most of this result follows from Theorem 5 and the discussion just above the statement of Theorem 9. The only remaining observation to make is that the inequality

\[\text{ord}(\lambda y_{r,k} z_{s,k}) \geq \frac{i_r + i'_s}{p}\]
is equivalent to
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' \geq \frac{i_r + i'_s}{p}, \]
and that follows from
\[ \text{ord}(u_{r,k} - 1) + \text{ord}(u_{k,s} - 1) - e' > \text{ord}(u_{r,s}) \]
and the inequalities
\[ \text{ord}(u_{r,s}) \geq \frac{i_r}{p} + \mu'_s \]
and
\[ \mu_s \leq i_s, \]
for then
\[ \frac{i_r}{p} + \mu'_s \geq \frac{i_r}{p} + i'_s \geq \frac{i_r}{p} + \frac{i'_s}{p}. \]
\[ \square \]

**Remark 10.** The extra inequalities
\[ (2p - 1)\text{ord}(\lambda y_{r,k}) \geq (p - 1)e' + i_r + i'_s \]
\[ (2p - 1)\text{ord}(\lambda y_{k,s}) \geq (p - 1)e' + i_r + i'_s \]
of Theorem 9 impose a mild extra restriction on the orders of the elements of Y beyond the inequalities of Theorem 5. From the inequalities of Theorem 5 we have
\[ \text{ord}(u_{r,k}) \geq \frac{i_r}{p} + \mu'_k \]
\[ \text{ord}(u_{k,s}) \geq \delta_k + \frac{i'_s}{p} \]
and so
\[ \text{ord}(u_{r,k}) + \text{ord}(u_{k,s}) \geq \frac{i_r}{p} + \mu'_k + \frac{i'_s}{p} \]
\[ \geq e' + \frac{i_r}{p} + \frac{i'_s}{p}. \]
\[ (*) \]
The additional inequalities of Theorem 9 may be restated as
\[ \text{ord}(u_{r,k}) \geq \frac{(p - 1)e'}{2p - 1} + \frac{i_r + i'_s}{2p - 1} \]
\[ \text{ord}(u_{k,s}) \geq \frac{(p - 1)e'}{2p - 1} + \frac{i_r + i'_s}{2p - 1}. \]
which, when added, yield

$$\text{ord}(u_{r,k}) + \text{ord}(u_{k,s}) \geq 2\frac{(p-1)e'}{2p-1} + 2 \frac{i_r + i'_s}{2p-1}$$

an inequality that follows from the inequality (*) derived from those of Theorem 5, since

$$e' + \frac{i_r}{p} + \frac{i'_s}{p} > 2\frac{(p-1)e'}{2p-1} + 2 \frac{i_r + i'_s}{2p-1}.$$ 

Remark 11. In constructing examples, we must satisfy a collection of linear inequalities on the valuations of \(\{u_{r,s} - 1\}\). The simplex algorithm is quite effective. Our examples do not need to satisfy any extremal property, but to fit the problem into a simplex program, it is convenient to specify an objective function to minimize. The following computations were done in Maple (TM) [Maple].

Let \(n = 5, p = 3, e' = 540\). Let

\((i_1, \ldots, i_5) = (360, 300, 240, 180, 120)\).

First choose the objective function to be the sum

\[ S = \sum_{r>s} \text{ord}(u_{r,s} - 1). \]

(i) An example minimizing \(S\) that satisfies the inequalities of Theorem 3 has a matrix \((\text{ord}(u_{r,s} - 1))\) of valuations as follows:

\[
\begin{pmatrix}
- & - & - & - & - \\
480 & - & - & - & - \\
420 & 480 & - & - & - \\
240 & 300 & 360 & - & - \\
220 & 280 & 340 & 520 & -
\end{pmatrix}.
\]

(ii) An example minimizing \(S\) that satisfies the inequalities of Theorem 5 (and of Theorem 9) has a matrix \((\text{ord}(u_{r,s} - 1))\) of valuations as follows:

\[
\begin{pmatrix}
- & - & - & - & - \\
528 & - & - & - & - \\
468 & 480 & - & - & - \\
288 & 300 & 360 & - & - \\
276 & 288 & 348 & 528 & -
\end{pmatrix}.
\]

Since \(S\) for Example (i) is smaller than for Example (ii), Example (i) is a Hopf order arising from Theorem 3 that does not have the form of the Hopf orders constructed in Theorem 5.

If we let \(S_5 = \sum_{s=1}^{4} \text{ord}(u_{5,s} - 1)\), then
(iii) an example that satisfies the inequalities of Theorem 9 and minimizes $S_5$ has a matrix $(\text{ord}(u_{r,s} - 1))$ of valuations as follows:

$$
\begin{pmatrix}
- & - & - & - & - \\
540 & - & - & - & - \\
520 & 540 & - & - & - \\
500 & 520 & 540 & - & - \\
276 & 288 & 340 & 400 & - \\
\end{pmatrix}.
$$

(iv) an example that satisfies the inequalities of Theorem 5 and minimizes $S_5$ has matrix:

$$
\begin{pmatrix}
- & - & - & - & - \\
540 & - & - & - & - \\
460 & 480 & - & - & - \\
400 & 420 & 480 & - & - \\
220 & 280 & 340 & 400 & - \\
\end{pmatrix}.
$$

Since $S_5$ is smaller for Example (iv) than for Example (iii), we have in Example (iii) a Hopf order that satisfies the inequalities of Theorem 5 but not the additional inequalities of Theorem 9. Thus not all of the Hopf orders constructed using Theorem 5 can be described using the truncated exponential method of Theorem 9.

The results in this paper extend and refine results from the second author’s Ph. D. thesis [Sm97].

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References


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