Q_K SPACES VIA HIGHER ORDER DERIVATIVES

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ABSTRACT. We characterize the Möbius invariant Q_K spaces in terms of higher order derivatives. Our methods are new even in the case of Q_p spaces.

1. INTRODUCTION

One of the classical topics in complex analysis is the study of Möbius invariant function spaces in the unit disk $\mathbb{D}$, namely, spaces of analytic functions equipped with a norm that is invariant under the action of Möbius maps. Examples of such spaces include the familiar disk algebra, H^∞, the Dirichlet space, the Bloch space, and BMOA.

A particular class of Möbius invariant function spaces, the so-called Q_p spaces, has attracted a lot of attention in recent years. More specifically, for any $0 \le p < \infty$, Q_p consists of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{Q_p}^2 = \sup_{\varphi} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi(z)|^2)^p \, dA(z) < \infty,$$

where dA is area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$, and the supremum is taken over $\varphi \in \mathrm{Aut}(\mathbb{D})$, the group of Möbius maps of the unit disk $\mathbb{D}$. The Möbius invariance of the Q_p norm is then a consequence of the well-known Möbius invariance of the Dirichlet integral. A good summary of recent research on Q_p spaces is Xiao's monograph [10].

Since every Möbius map φ can be written as $\varphi(z) = e^{i\theta}\varphi_a(z)$, where θ is real and

$$\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$$

is the Möbius map of the unit disk that interchanges the points 0 and a, we can also write

$$\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p \, dA(z).$$

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Three special cases are worth mentioning. When $p = 0$, Q_p becomes the Dirichlet space; when $p = 1$, Q_p coincides with BMOA; and when $p > 1$, Q_p is just the Bloch space.

There are a number of ways we can further generalize the Q_p spaces; see [11] and [12] for example. In this paper we are concerned with a particular type of generalization, the so-called Q_K spaces. Thus for any nonnegative, Lebesgue measurable function K on $(0, 1]$, we consider the space Q_K consisting of all analytic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2)\, dA(z) < \infty.$$

Clearly, if $K(t) = t^p$, then $Q_K = Q_p$. It is also clear that Q_K is Möbius invariant in the sense that $\|f \circ \varphi\|_{Q_K} = \|f\|_{Q_K}$ whenever $f \in Q_K$ and $\varphi \in \mathrm{Aut}(\mathbb{D})$. See [1] for a general exposition on Möbius invariant function spaces.

The study on Q_K spaces has mainly been on understanding the relationship between the properties of K and the resulting space Q_K. See [5] [6] [9] for example. Each space Q_K contains all constant functions. Since Q_K is Möbius invariant, it contains a nonconstant function (in this case, we say that Q_K is non-trivial) if and only if it contains all polynomials; see [1]. Therefore, Q_K is non-trivial if and only if the coordinate function z is in Q_K, that is,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2)\, dA(z) < \infty.$$

Making an obvious change of variables in the above integral and simplifying the result using polar coordinates, we see that Q_K is non-trivial if and only if

$$\sup_{0<s<1} (1 - s)^2 \int_0^1 \frac{K(1 - r)\, dr}{(1 - rs)^3} < \infty.$$

In the rest of the paper, we always assume that K satisfies this condition. We further assume that K is continuous and non-decreasing on $(0, 1]$. We then extend the domain of K to $(0, \infty)$ by setting $K(t) = K(1)$ when $t > 1$, so that K becomes a continuous and non-decreasing function on $(0, \infty)$.

The purpose of this paper is to characterize the Q_K spaces in terms of higher order derivatives. The corresponding problem for the Q_p spaces is studied in [2]. The main diffficulty for the Q_K case is finding the right condition on K that will ensure the higher order derivative characterization. We find out that a previously defined, familiar conditon on K is enough for us here. More specifically, we need the auxiliary function

$$\varphi_K(s) = \sup_{0<t\le 1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty. \tag{1}$$

Since K is non-decreasing, the function φ_K is also non-decreasing. Furthermore, we always have $\varphi_K(s) \le 1$ for $0 < s \le 1$ and $\varphi_K(s) \ge 1$ for $s \ge 1$.

We obtain two conditions on the function $\varphi_K(s)$, one near the point $s = 0$ and the other near $s = \infty$, that will enure a higher order derivative characterization for the spaces Q_K.

Theorem A. *Suppose there exists some $p < 2$ such that*

$$\int_1^\infty \frac{\varphi_K(s)}{s^p}\, ds < \infty.$$

Then for any positive integer n, an analytic function f in $\mathbb{D}$ belongs to Q_K if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} K(1-|\varphi_a(z)|^2)\, dA(z) < \infty.$$

Theorem B. *Suppose the function K satisfies $\varphi_K(2) < \infty$ and*

$$\int_0^1 \frac{\varphi_K(s)}{s}\, ds < \infty.$$

Then for any positive integer n, an analytic function f in $\mathbb{D}$ belongs to Q_K if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} K(1-|\varphi_a(z)|^2)\, dA(z) < \infty.$$

In particular, we obtain new proofs of the higher order derivative characterization for Q_p spaces. One of our methods also works for Q_K spaces defined on the open unit ball in $\mathbb{C}^n$.

It is easy to see that the condition $\varphi_K(2) < \infty$ in Theorem B can be replaced by $\varphi_K(s) < \infty$ for some $s > 1$. Obviously, if the integral condition in Theorem A is satisfied, then $\varphi_K(s) < \infty$ for every $s > 0$.

Our definition of Q_K here is based on $K(1-|\varphi_a(z)|^2)$. There is a slightly different definition of Q_K in the literature that is based on $K(G(a,z))$, where G is the Green function of the unit disk. However, it has been known that the two definitions are essentially equivalent; see [3] or Theorem 3.1 of [6]. In particular, the definition Q_p based on $(1-|\varphi_a(z)|^2)^p$ is equivalent to the definition of Q_p based on $G(a,z)^p$; see Theorem 1.1.1 of [10]. From a technical view point, the expression $K(1-|\varphi_a(z)|^2)$ is easier to manipulate than the expression $K(G(a,z))$.

2. PRELIMINARIES

One of the tools for our analysis is the classical Schur's test which concerns the boundedness of integral operators with positive kernels on L^p spaces.

Let (X, μ) be a measure space. Consider integral operators of the form

$$Tf(x) = \int_X H(x,y) f(y)\, d\mu(y), \tag{2}$$

where H is a nonnegative, measurable function on $X \times X$.

Lemma 1. *Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. If there exists a positive, measurable function h on X such that*

$$\int_X H(x,y) h(y)^q\, d\mu(y) \le C h(x)^q$$

for almost all $x \in X$ and

$$\int_X H(x,y) h(x)^p\, d\mu(x) \le C h(y)^p$$

for almost all $y \in X$, where C is a positive constant, then the integral operator T defined in (2) is bounded on $L^p(X, d\mu)$. Furthermore, the norm of T on $L^p(X, d\mu)$ does not exceed the constant C.

Proof. This is classical. See Theorem 3.2.2 of [13] for example. □

We are going to use the following special case.

Lemma 2. *If there exists a positive, measurable function h on X such that*

$$\int_X H(x,y) h(y)\, d\mu(y) \le C h(x)$$

for almost all x and

$$\int_X H(x,y) h(x)\, d\mu(x) \le C h(y)$$

for almsot all y, then the operator T defined in (2) is bounded on $L^2(X, d\mu)$, and the norm of T on $L^2(X, d\mu)$ is less than or equal to the constant C.

Proof. This clearly follows from Lemma 1. □

Note that the lemma above not only tells when an integral operator is bounded, it also gives an estimate on the norm of the operator. This norm estimate will be essential for our analysis later on.

Our strategy is to compare the integrals

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2)\, dA(z)$$

and

$$\int_{\mathbb{D}} |f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)\,dA(z),$$

where a is any fixed point in $\mathbb{D}$. We do this using Schur's test, and show that the comparison can be made uniformly for $a \in \mathbb{D}$, provided that the function K satisfies the integral condition in Theorems A. An alternative approach is presented in Section 5 based on local properties of the function φ_K near the origin.

The following estimate will be needed several times later on.

Lemma 3. *Suppose $t > -1$. If $s > 0$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t\,dA(w)}{|1-z\overline{w}|^{2+t+s}} \le \frac{C}{(1-|z|^2)^s}$$

for all $z \in \mathbb{D}$. If $s < 0$, then there exists a constant $C > 0$ such that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t\,dA(w)}{|1-z\overline{w}|^{2+t+s}} \le C$$

for all $z \in \mathbb{D}$.

Proof. This is well known. See Lemma 4.2.2 of [13] for example. □

We also need the following estimate.

Lemma 4. *If $s > 1$, then there exists a constant $C > 0$ such that*

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^s} \le \frac{C}{(1-|z|^2)^{s-1}}$$

for all $z \in \mathbb{D}$. If $s < 1$, then there exists a constant $C > 0$ such that

$$\int_0^{2\pi} \frac{d\theta}{|1-ze^{-i\theta}|^s} \le C$$

for all $z \in \mathbb{D}$.

Proof. This is also well known. See Section 4.6 of [4] or Theorem 1.7 of [7]. □

3. Raising the order of derivative

Let a be a point in $\mathbb{D}$ and fix a positive integer n. Define a positive measure μ_a on the unit disk by

$$d\mu_a(z) = K(1-|\varphi_a(z)|^2)\,dA(z) \tag{3}$$

and consider the integral operator

$$T_a f(z) = \int_{\mathbb{D}} H_a(z,w) f(w)\,d\mu_a(w), \tag{4}$$

where

$$H_a(z,w)=\frac{(1-|z|^2)^n(1-|w|^2)^\alpha}{|1-z\overline{w}|^{2+n+\alpha}K(1-|\varphi_a(w)|^2)} \tag{5}$$

is a positive integral kernel and α is a sufficiently large constant.

Our goal in this section is to show that the operator T_a is bounded on $L^2(\mathbb{D},d\mu_a)$ and the norm of T_a on $L^2(\mathbb{D},d\mu_a)$ is bounded for $a\in\mathbb{D}$. As a consequence, we will show that every function in Q_K satisfies an estimate in terms of higher oder derivatives.

Theorem 5. *Suppose there exists some $N\in(0,2n+2)$ such that*

$$\int_1^\infty \frac{\varphi_K(s)}{s^N}\,ds<\infty. \tag{6}$$

Then for α sufficiently large, the operator T_a defined in (4) is bounded on $L^2(\mathbb{D},d\mu_a)$. Moreover, there exists a constant $C>0$, independent of a (but dependent on α), such that

$$\int_{\mathbb{D}}|T_af(z)|^2\,d\mu_a(z)\le C\int_{\mathbb{D}}|f(z)|^2\,d\mu_a(z)$$

for all $f\in L^2(\mathbb{D},d\mu_a)$.

Proof. We prove the theorem using Lemma 2. To this end, we fix a constant σ such that

$$N-(n+2)\le\sigma<n. \tag{7}$$

This is possible, because the conditon $N<2n+2$ gives $N-(n+2)<n$. We also fix a constant α such that

$$\alpha>n+2\sigma+1,\quad \alpha+\sigma>-1,\quad \alpha>0. \tag{8}$$

We now verify the conditions in Lemma 2 using the function

$$h(z)=(1-|z|^2)^\sigma,\qquad z\in\mathbb{D}.$$

First, for any $z\in\mathbb{D}$ we have

$$\int_{\mathbb{D}}H_a(z,w)h(w)\,d\mu_a(w)=(1-|z|^2)^n\int_{\mathbb{D}}\frac{(1-|w|^2)^{\alpha+\sigma}\,dA(w)}{|1-z\overline{w}|^{2+(\alpha+\sigma)+(n-\sigma)}}.$$

Since $\alpha+\sigma>-1$ and $n-\sigma>0$, an application of Lemma 3 shows that there exists a constant $C>0$, independent of a, such that

$$\int_{\mathbb{D}}H_a(z,w)h(w)\,d\mu_a(w)\le Ch(z)$$

for all $z\in\mathbb{D}$.

Next, for any $w \in \mathbb{D}$ we consider the integral

$$I_a(w) = \int_{\mathbb{D}} H_a(z, w) h(z)\, d\mu_a(z).$$

It is clear that

$$I_a(w) = \frac{(1-|w|^2)^\alpha}{K(1-|\varphi_a(w)|^2)} \int_{\mathbb{D}} \frac{(1-|z|^2)^{n+\sigma} K(1-|\varphi_a(z)|^2)\, dA(z)}{|1-z\overline{w}|^{2+n+\alpha}}.$$

We make a change of variables $z = \varphi_w(u)$ in the above expression and use the fact that there exists a unimodulus constant $e^{i\theta}$ such that

$$\varphi_a \circ \varphi_w(u) = e^{i\theta} \varphi_\lambda(u),$$

where $\lambda = \varphi_w(a)$. After the result is simplified, we obtain

$$I_a(w) = \frac{(1-|w|^2)^\sigma}{K(1-|\lambda|^2)} \int_{\mathbb{D}} \frac{(1-|u|^2)^{n+\sigma} K(1-|\varphi_\lambda(u)|^2)\, dA(u)}{|1-w\overline{u}|^{2+n+2\sigma-\alpha}}.$$

Since

$$1-|\varphi_\lambda(u)|^2 = \frac{(1-|\lambda|^2)(1-|u|^2)}{|1-\overline{\lambda}u|^2},$$

it follows from the definition of the auxiliary function φ_K that

$$K(1-|\varphi_\lambda(u)|^2) \le \varphi_K\left(\frac{1-|u|^2}{|1-\overline{\lambda}u|^2}\right) K(1-|\lambda|^2).$$

Therefore,

$$I_a(w) \le h(w) \int_{\mathbb{D}} \frac{(1-|u|^2)^{n+\sigma}}{|1-w\overline{u}|^{2+n+2\sigma-\alpha}} \varphi_K\left(\frac{1-|u|^2}{|1-\overline{\lambda}u|^2}\right) dA(u).$$

Since the function φ_K is non-decreasing, and since

$$\frac{1-|u|^2}{|1-\overline{\lambda}u|^2} \le \frac{1-|u|^2}{(1-|u|)^2} = \frac{1+|u|}{1-|u|},$$

we must have

$$\varphi_K\left(\frac{1-|u|^2}{|1-\overline{\lambda}u|^2}\right) \le \varphi_K\left(\frac{1+|u|}{1-|u|}\right),$$

so

$$I_a(w) \le h(w) \int_{\mathbb{D}} \frac{(1-|u|^2)^{n+\sigma}}{|1-w\overline{u}|^{2+n+2\sigma-\alpha}} \varphi_K\left(\frac{1+|u|}{1-|u|}\right) dA(u).$$

Write the above integral in polar coordinates, use the condition that

$$2+n+2\sigma-\alpha < 1,$$

which follows from the first assumption in (8), and apply Lemma 4. We find a constant $C > 0$, independent of a, such that

$$I_a(w) \le Ch(w) \int_0^1 (1-r^2)^{n+\sigma} \varphi_K\left(\frac{1+r}{1-r}\right) r\, dr.$$

It is clear that we can find another positive constant C, independent of a, such that

$$I_a(w) \le Ch(w) \int_0^1 \left(\frac{1-r}{1+r}\right)^{n+\sigma} \varphi_K\left(\frac{1+r}{1-r}\right) dr.$$

Now make a change of variables according to $s = \frac{1+r}{1-r}$. The result is

$$\int_{\mathbb{D}} H_a(z,w)h(z)\,d\mu_a(z) \le Ch(w) \int_1^\infty \frac{\varphi_K(s)\,ds}{s^{n+2+\sigma}}.$$

Recall from (7) that $n+2+\sigma \ge N$, so the integral above converges, and we obtain another positive constant C, independent of a, such that

$$\int_{\mathbb{D}} H_a(z,w)h(z)\,d\mu_a(z) \le Ch(w)$$

for all $w \in \mathbb{D}$.

In view of Lemma 2, the proof of the theorem is now complete. □

Corollary 6. *Suppose K satisfies condition (6) and $f \in Q_K$. Then*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n+1)}(z)|^2(1-|z|^2)^{2n} K(1-|\varphi_a(z)|^2)\,dA(z) < \infty. \tag{9}$$

Proof. Choose σ and α according to the proof of Theorem 5. Since Q_K is Möbius invariant and the Bloch space is the maximal Möbius invariant space (see [8]), we must have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

This together with the assumption $\alpha > 0$ in (8) shows that we have the following integral representation:

$$f'(z) = (\alpha+1)\int_{\mathbb{D}} \frac{f'(w)(1-|w|^2)^\alpha\,dA(w)}{(1-z\overline{w})^{2+\alpha}}, \qquad z \in \mathbb{D};$$

see Corollary 1.5 in [7]. Differentiating under the integral n times and multiplying the result by $(1-|z|^2)^n$, we obtain

$$(1-|z|^2)^n f^{(n+1)}(z) = C\int_{\mathbb{D}} \frac{(1-|z|^2)^n(1-|w|^2)^\alpha\overline{w}^n f'(w)\,dA(w)}{(1-z\overline{w})^{2+\alpha+n}},$$

where C is a positive constant depending only on α and n. In particular,

$$(1-|z|^2)^n|f^{(n+1)}(z)| \le C\int_{\mathbb{D}} H_a(z,w)|f'(w)|\,d\mu_a(w).$$

By Theorem 5, there exists a constant $C>0$, independent of a, such that

$$\int_{\mathbb{D}}(1-|z|^2)^{2n}|f^{(n+1)}(z)|^2\,d\mu_a(z) \le C\int_{\mathbb{D}}|f'(z)|^2\,d\mu_a(z)$$

for all $a \in \mathbb{D}$. Since

$$d\mu_a(z) = K(1 - |\varphi_a(z)|^2)\, dA(z),$$

and since the above estimate holds for all $a \in \mathbb{D}$, taking the supremum over $a \in \mathbb{D}$ leads to the desired estimate (9). □

4. LOWERING THE ORDER OF DERIVATIVE

In this section we show that under a more restrictive condition than (6), the condition in (9) implies that $f \in Q_K$. To this end, we consider integral operators of the form

$$S_a f(z) = \int_{\mathbb{D}} L_a(z,w) f(w)\, d\mu_a(w), \tag{10}$$

where $d\mu_a$ is defined in (3) and

$$L_a(z,w) = \frac{(1-|w|^2)^\alpha}{|1 - z\overline{w}|^{2+\alpha} K(1 - |\varphi_a(w)|^2)}.$$

Again, α is a sufficiently large constant.

Theorem 7. *Suppose K satisfies the integral condition*

$$\int_1^\infty \frac{\varphi_K(s)}{s^{2-\epsilon}}\, ds < \infty \tag{11}$$

for some $\epsilon \in (0,1)$. Then for any $\alpha > 1$, the operator S_a defined in (10) is bounded on $L^2(\mathbb{D}, d\mu_a)$. Moreover, there exists a constant $C > 0$, independent of a, such that

$$\int_{\mathbb{D}} |S_a f(z)|^2\, d\mu_a(z) \le C \int_{\mathbb{D}} |f(z)|^2\, d\mu_a(z)$$

for all $f \in L^2(\mathbb{D}, d\mu_a)$.

Proof. We use the function

$$h(z) = (1 - |z|^2)^{-\epsilon}, \qquad z \in \mathbb{D},$$

and apply Lemma 2 to show the boundedness of S_a on $L^2(\mathbb{D}, d\mu_a)$.

First, for any $z \in \mathbb{D}$, we have

$$\int_{\mathbb{D}} L_a(z,w) h(w)\, d\mu_a(w) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha-\epsilon}\, dA(w)}{|1 - z\overline{w}|^{2+(\alpha-\epsilon)+\epsilon}}.$$

Since $\alpha - \epsilon > -1$, an application of Lemma 3 yields a positive constant $C > 0$, independent of a, such that

$$\int_{\mathbb{D}} L_a(z,w) h(w)\, d\mu_a(w) \le C h(z)$$

for all $z \in \mathbb{D}$.

Next, for any $w\in\mathbb{D}$, we consider the integral

$$J_a(w)=\int_{\mathbb{D}}L_a(z,w)h(z)\,d\mu_a(z).$$

It is clear that

$$J_a(w)=\frac{(1-|w|^2)^\alpha}{K(1-|\varphi_a(w)|^2)}\int_{\mathbb{D}}\frac{(1-|z|^2)^{-\epsilon}K(1-|\varphi_a(z)|^2)\,dA(z)}{|1-z\overline{w}|^{2+\alpha}}.$$

Just like in the proof of Theorem 5, we make the change of variables $z=\varphi_w(u)$ and simplify the result to obtain

$$J_a(w)=\frac{h(w)}{K(1-|\lambda|^2)}\int_{\mathbb{D}}\frac{(1-|u|^2)^{-\epsilon}K(1-|\varphi_\lambda(u)|^2)\,dA(u)}{|1-w\overline{u}|^{2-2\epsilon-\alpha}},$$

where $\lambda=\varphi_w(a)$. Duplicating the corresponding part of the proof of Theorem 5, we obtain a positive constant C, independent of a, such that

$$J_a(w)\le Ch(w)\int_1^\infty\frac{\varphi_K(s)\,ds}{s^{2-\epsilon}}.$$

By condition (11), there exists a constant $C>0$, independent of a, such that

$$\int_{\mathbb{D}}L_a(z,w)h(z)\,d\mu_a(z)\le Ch(w)$$

for all $w\in\mathbb{D}$.

An application of Lemma 2 then completes the proof of the theorem. □

Corollary 8. *Suppose K satisfies condition (11) and f satisfies condition (9). Then f belongs to Q_K.*

Proof. Since every polynomial belongs to Q_K and every polynomial satisfies condition (9), by substracting a Taylor polynomial from f we may as well assume that

$$f(0)=f'(0)=\cdots=f^{(n)}(0)=0.$$

We use this and integrate both sides of the reproducing formula (see Corollary 1.5 of [7] for example)

$$f^{(n+1)}(z)=(n+\alpha+1)\int_{\mathbb{D}}\frac{f^{(n+1)}(w)(1-|w|^2)^n(1-|w|^2)^\alpha\,dA(w)}{(1-z\overline{w})^{2+n+\alpha}}$$

n times to produce

$$f'(z)=\int_{\mathbb{D}}\frac{h(z,w)f^{(n+1)}(w)(1-|w|^2)^n(1-|w|^2)^\alpha\,dA(w)}{(1-z\overline{w})^{2+\alpha}},$$

where $h(z,w)$ is a bounded function in z and w (if we chose α to be a positive integer, then $h(z,w)$ is a polynomial in z and $\overline{w}$). In particular, there exists a constant $C>0$, independent of a, such that

$$|f'(z)| \le C \int_{\mathbb{D}} L_a(z,w)|f^{(n+1)}(w)(1-|w|^2)^n|\,d\mu_a(w).$$

By Theorem 7, there exists a constant $C>0$, independent of a, such that

$$\int_{\mathbb{D}} |f'(z)|^2\,d\mu_a(z) \le C\int_{\mathbb{D}} |f^{(n+1)}(z)|^2(1-|z|^2)^{2n}\,d\mu_a(z)$$

for all $a\in\mathbb{D}$. Go back to

$$d\mu_a(z) = K(1-|\varphi_a(z)|^2)\,dA(z)$$

and take the supremum over all $a\in\mathbb{D}$. We deduce from condition (9) that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\,dA(z)<\infty,$$

or $f\in Q_K$. $\square$

Combining Corollaries 6 and 8, we obtain Theorem A, which we restate as follows.

Theorem 9. *Suppose K satisfies condition (11). Then for any positive integer n and any analytic function f in $\mathbb{D}$ we have $f\in Q_K$ if and only if*

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)\,dA(z)<\infty.$$

Up to this point, we have not used the continuity of the weight function K. The continuity of K will be needed in the next section along with some other regularity conditions of K near the origin.

5. Another Approach

In this section we present a different approach to the problem of characterizaing Q_K functions in terms of higher order derivatives. This method requires the function $K(t)$ to be more regular near the point $t=0$.

Lemma 10. *Suppose the function K satisfies the following two conditions:*

$$\varphi_K(2)<\infty,\qquad \int_0^1 \frac{\varphi_K(s)}{s}\,ds<\infty. \tag{12}$$

Then the function

$$K_*(t)=\int_0^t \frac{K(s)}{s}\,ds,\qquad 0<t\le 1,$$

is comparable to K, that is, there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{K_*(t)}{K(t)} \leq C \tag{13}$$

for all $t > 0$. Here again, $K_*(t) = K_*(1)$ when $t > 1$.

Proof. See [6]. □

Theorem 11. *Suppose K satisfies the conditions in (12). Then for any $\alpha > -1$ the integral*

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha K(1-|z|^2)\, dA(z)$$

is comparable to the integral

$$|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+2} K(1-|z|^2)\, dA(z),$$

where f is analytic in $\mathbb{D}$.

Proof. Fix an analytic function f in $\mathbb{D}$ with Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad z \in \mathbb{D}.$$

Without loss of generality we may assume that f is analytic on the closed unit disk. Otherwise, we work with $f(\rho z)$ first, where $0 < \rho < 1$, then take limits as $\rho \to 1$ and use Lebesgue's monotone convergence theorem.

We consider the integral

$$I_\alpha(f) = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha K(1-|z|^2)\, dA(z).$$

Integration in polar coordinates shows that

$$I_\alpha(f) = \int_0^1 (1-r)^\alpha K(1-r) \left(\sum_{k=0}^{\infty} |a_k|^2 r^k \right) dr. \tag{14}$$

We write

$$K(1-r)\, dr = -(1-r)\, dK_*(1-r)$$

and integrate in parts to obtain

$$\begin{aligned} I_\alpha(f) &= K_*(1)|f(0)|^2 \\ &\quad -(\alpha+1) \int_0^1 (1-r)^\alpha K_*(1-r) \left(\sum_{k=0}^{\infty} |a_k|^2 r^k \right) dr \\ &\quad + \int_0^1 (1-r)^{\alpha+1} K_*(1-r) \left(\sum_{k=1}^{\infty} |a_k|^2 k r^{k-1} \right) dr. \end{aligned}$$

Therefore,

$$I_\alpha(f) + (\alpha+1)\int_0^1 (1-r)^\alpha K_*(1-r)\left(\sum_{k=0}^\infty |a_k|^2 r^k\right)\,dr \tag{15}$$

is equal to

$$K_*(1)|f(0)|^2 + \int_0^1 (1-r)^{\alpha+1} K_*(1-r)\left(\sum_{k=1}^\infty |a_k|^2 k r^{k-1}\right)\,dr. \tag{16}$$

Combining (13) with (14), (15), and (16), we conclude that

$$I_\alpha(f) \sim |f(0)|^2 + \int_0^1 (1-r)^{\alpha+1} K(1-r)\left(\sum_{k=1}^\infty |a_k|^2 k r^{k-1}\right)\,dr. \tag{17}$$

Similarly, integration by parts along with (13) shows that the integral

$$J_\alpha(f) = \int_0^1 (1-r)^{\alpha+1} K(1-r)\left(\sum_{k=1}^\infty |a_k|^2 k r^{k-1}\right)\,dr$$

is comparable to

$$|f'(0)|^2 + \int_0^1 (1-r)^{\alpha+2} K(1-r)\left(\sum_{k=2}^\infty |a_k|^2 k(k-1) r^{k-2}\right)\,dr.$$

Since $k(k-1)$ is comparable to k^2 and, with the help of polar coordinates, the integral

$$\int_0^1 (1-r)^{\alpha+2} K(1-r)\left(\sum_{k=2}^\infty |a_k|^2 k^2 r^{k-2}\right)\,dr$$

can be written as

$$\int_{\mathbb{D}} (1-|z|^2)^{\alpha+2} K(1-|z|^2)\frac{|f'(z)-f'(0)|^2}{|z|}\,dA(z),$$

we see that $I_\alpha(f)$ is comparable to

$$|f(0)^2 + |f'(0)|^2 + \int_{\mathbb{D}} (1-|z|^2)^{\alpha+2} K(1-|z|^2)\frac{|f'(z)-f'(0)|^2}{|z|}\,dA(z).$$

A standard argument using the sub-mean-value property then shows that $I_\alpha(f)$ is comparable to

$$|f(0)|^2 + \int_{\mathbb{D}} (1-|z|^2)^{\alpha+2} K(1-|z|^2)|f'(z)|^2\,dA(z),$$

completing the proof of the theorem. □

Lemma 12. *If f satisfies condition (9), then f belongs to the Bloch space.*

Proof. After a change of variables, condition (9) becomes

$$(1-|a|^2)^{2(n+1)}\int_{\mathbb{D}}\left|\frac{f^{(n+1)}(\varphi_a(z))}{(1-\overline{a}z)^{2(n+1)}}\right|^2(1-|z|^2)^{2n}K(1-|z|^2)\,dA(z)\le C,$$

where C is a positive constant independent of $a\in\mathbb{D}$. We integrate in polar coordinates and use the sub-mean-value property at $z=0$ to obtain

$$(1-|a|^2)^{2(n+1)}|f^{(n+1)}(a)|^2\int_{\mathbb{D}}(1-|z|^2)^{2n}K(1-|z|^2)\,dA(z)\le C.$$

This shows that

$$\sup_{a\in\mathbb{D}}(1-|a|^2)^{n+1}|f^{(n+1)}(a)|<\infty.$$

According to Theorem 5.1.5 of [13], f is a Bloch function. □

We now prove the second main result of the paper, Theorem B, which we restate as follows.

Theorem 13. *Suppose K satisfies the conditions in (12) and f is analytic in $\mathbb{D}$. Then $f\in Q_K$ if and only if f satisfies condition (9).*

Proof. By Lemma 12 and the fact that Q_K is contained in the Bloch space, we may as well assume that f is in the Bloch space.

Let n be a positive integer and consider the integral

$$I_n(f,a)=\int_{\mathbb{D}}|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)\,dA(z).$$

By a change of variables,

$$I_n(f,a)=\int_{\mathbb{D}}|g(z)|^2(1-|z|^2)^{2n-2}K(1-|z|^2)\,dA(z),$$

where

$$g(z)=f^{(n)}(\varphi_a(z))(1-|a|^2)^n(1-\overline{a}z)^{-2n}.$$

According to Theorem 11, $I_n(f,a)$ is comparable (uniform in a) to

$$(1-|a|^2)^{2n}|f^{(n)}(a)|^2+J_n(f,a),$$

where

$$J_n(f,a)=\int_{\mathbb{D}}|g'(z)|^2(1-|z|^2)^{2n}K(1-|z|^2)\,dA(z).$$

Since f is in the Bloch space, we see that

$$\sup_{a\in\mathbb{D}}I_n(f,a)<\infty$$

if and only if

$$\sup_{a\in\mathbb{D}}J_n(f,a)<\infty.$$

We differentiate g using the product rule and write

$$g'(z) = 2n\overline{a}h_1(z) - h_2(z),$$

where

$$h_1(z) = f^{(n)}(\varphi_a(z))(1-|a|^2)^n(1-\overline{a}z)^{-2n-1}$$

and

$$h_2(z) = f^{(n+1)}(\varphi_a(z))(1-|a|^2)^{n+1}(1-\overline{a}z)^{-2(n+1)}.$$

After a change of variables, the integral

$$\int_{\mathbb{D}} |h_2(z)|^2(1-|z|^2)^{2n}K(1-|z|^2)\,dA(z)$$

becomes

$$I_{n+1}(f,a) = \int_{\mathbb{D}} |f^{(n+1)}(z)|^2(1-|z|^2)^{2n}K(1-|\varphi_a(z)|^2)\,dA(z).$$

On the other hand, the integral

$$H(a) = \int_{\mathbb{D}} |h_1(z)|^2(1-|z|^2)^{2n}K(1-|z|^2)\,dA(z)$$

can be written as

$$H(a) = \int_{\mathbb{D}} (1-|\varphi_a(z)|^2)^{2n}|f^{(n)}(\varphi_a(z))|^2\frac{K(1-|z|^2)}{|1-\overline{a}z|^2}\,dA(z).$$

Since f is in the Bloch space, there is a positive constant C such that

$$H(a) \le C\int_{\mathbb{D}} \frac{K(1-|z|^2)}{|1-\overline{a}z|^2}\,dA(z).$$

Integrating in polar coordinates and applying Lemma 10, we find out that

$$H(a) \le C'\int_0^1 \frac{K(1-r)}{1-r}\,dr = C'\int_0^1 \frac{K(s)}{s}\,ds \le C''K(1),$$

where C' and C'' are positive constants independent of a. Combining this with the triangle inequality, we deduce that the integral $J_n(f,a)$ is bounded for $a\in\mathbb{D}$ if and only if the integral $I_{n+1}(f,a)$ is bounded for $a\in\mathbb{D}$. Therefore, $I_n(f,a)$ is bounded in a if and only if $I_{n+1}(f,a)$ is bounded in a. The theorem is then proved by induction. □

6. The Spaces $Q_{K,0}$

Let $Q_{K,0}$ denote the space of analytic functions f in $\mathbb{D}$ such that

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\, dA(z) = 0.$$

Since our results in Corollaries 6 and 8 are pointwise estimates at a, we also have the following little oh version of Theorem 9.

Theorem 14. *Suppose K satisfies condition (11). Then for any positive integer n and any analytic function f in $\mathbb{D}$ we have $f \in Q_{K,0}$ if and only if*

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} K(1-|\varphi_a(z)|^2)\, dA(z) = 0.$$

Carefully checking the proof of Theorem 13, we also obtain the following little oh version of Theorem 13.

Theorem 15. *Suppose the function K satisfies the conditions in (12). Then for any positive integer n, an analytic function f in $\mathbb{D}$ belongs to $Q_{K,0}$ if and only if*

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f^{(n)}(z)|^2 (1-|z|^2)^{2n-2} K(1-|\varphi_a(z)|^2)\, dA(z) = 0.$$

In many situations, the subspace $Q_{K,0}$ is simply the closure in Q_K of the set of polynomials. Examples include BMOA and the Bloch space. There are also cases when the polynomials are dense in Q_K. For example, if $K = 1$, then Q_K is the Dirichlet space; in this case, the polynomials are dense, but $Q_{K,0}$ as defined at the beginning of this section consists of just the constant functions.

We have obtained higher order derivative characterizations of Q_K based on the behavior of $\varphi_K(s)$ near $s = 0$ and near $s = \infty$. It would be interesting to find the "optimal conditions" on K that will ensure the higher order derivative description of Q_K spaces.

7. The Case of Q_p

If $K(t) = t^p$, then $\varphi_K(s) = s^p$, so condition (11) holds if and only if $p < 1$. However, in this particular case, it is not necessary to use the auxiliary function φ_K, and our proofs of Theorems 5 and 7 can be simplified, and the statements of these theorems can be improved to cover all $p > 0$.

In fact, even with the use φ_K, Theorem 5 remains true for Q_p whenever $p < 3$, because the constant $2n+2$ is at least 4.

In the proof of Theorem 7, if $K(t) = t^p$, then the integral $J_a(w)$ can be written as

$$J_a(w) = \frac{h(w)}{(1-|\lambda|^2)^p} \int_{\mathbb{D}} \frac{(1-|u|^2)^{-\epsilon}(1-|\varphi_\lambda(u)|^2)^p \, dA(u)}{|1-w\overline{u}|^{2-2\epsilon-\alpha}}.$$

A little manipulation gives

$$J_a(w) = h(w) \int_{\mathbb{D}} \frac{(1-|u|^2)^{p-\epsilon} \, dA(u)}{|1-w\overline{u}|^{2-2\epsilon-\alpha}|1-\overline{\lambda}u|^{2p}}.$$

We may assume that α was large enough so that $2-2\epsilon-\alpha \leq 0$. Then there exists a constant $C > 0$, independent of a, such that

$$J_a(w) \leq Ch(w) \int_{\mathbb{D}} \frac{(1-|u|^2)^{p-\epsilon} \, dA(u)}{|1-\overline{\lambda}u|^{2p}}.$$

If $p < 2$, we can find a small enough ϵ so that $p + \epsilon < 2$, and then an application of Lemma 3 shows that there exists a constant $C > 0$, independent of a, such that

$$J_a(w) \leq Ch(w), \qquad w \in \mathbb{D}.$$

In conclusion, our first method can be modified to work for Q_p whenever $p < 2$. Since Q_p is the Bloch space for all $p > 1$, this method covers all Q_p spaces. Also, since $\varphi_K(s) = s^p$ for $K(t) = t^p$, the conditions in (12) hold Q_p for all $p > 0$. Therefore, our second method works for all Q_p spaces as well.

REFERENCES

[1] J. Arazy, S. Fisher, and J. Peetre, Möbiuns invariant function spaces, *J. Reine Angew. Math.* **363** (1985), 110-145.

[2] R. Aulaskari, M. Nowak, and R. Zhao, The nth derivative characterization of the Möbius bounded Dirichlet space, *Bull. Australia Math. Soc.* **58** (1998), 43-56.

[3] R. Aulaskari, D. Stegenga, and J. Xiao, Some subclasses of BMOA and their characterization in terms of Carleson measures, *Rocky Mountain J. Math.* **26** (1996), 485-506.

[4] P. Duren, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[5] M. Essén and H. Wulan, On analytic and meromorphic functions and spaces of Q_K type, *Illinois J. Math.* **46** (2002), 1233-1258.

[6] M. Essén, H. Wulan, and J. Xiao, Function-theoretic aspects of Möbius invariant Q_K spaces, to appear in *J. Funct. Anal.*

[7] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer, New York, 2000.

[8] L. Rubel and R. Timoney, An extremal property of the Bloch space, *Proc. Amer. Math. Soc.* **75** (1979), 45-49.

[9] H. Wulan and P. Wu, Charaterization of Q_T spaces, *J. Math. Anal. Appl.* **254** (2001), 484-497.

[10] J. Xiao, *Holomorphic Q Classes*, Springer LNM **1767**, Berlin, 2001.

[11] R. Zhao, On a general family of function spaces, *Ann. Acad. Sci. Fenn. Math. Dissertationes* **105** (1996), 1-56.
[12] K. Zhu, A class of Möbius invariant function spaces, preprint, 2004.
[13] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

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