

COMPOSITION OPERATORS INDUCED BY SYMBOLS DEFINED ON A POLYDISK

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ABSTRACT. Suppose φ is a holomorphic mapping from the polydisk \mathbb{D}^m into the polydisk \mathbb{D}^n , or from the polydisk \mathbb{D}^m into the unit ball \mathbb{B}_n , we consider the action of the associated composition operator C_φ on Hardy and weighted Bergman spaces of \mathbb{D}^n or \mathbb{B}_n . We first find the optimal range spaces and then characterize compactness. As a special case, we show that if

$$\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z)), \quad z = (z_1, \dots, z_n),$$

is a holomorphic self-map of the polydisk \mathbb{D}^n , then C_φ maps $A_\alpha^p(\mathbb{D}^n)$ boundedly into $A_\beta^p(\mathbb{D}^n)$, the weight $\beta = n(2 + \alpha) - 2$ is best possible, and the operator

$$C_\varphi : A_\alpha^p(\mathbb{D}^n) \rightarrow A_\beta^p(\mathbb{D}^n)$$

is compact if and only if the function

$$\left[\prod_{k=1}^n (1 - |z_k|^2)^n \right] / \left[\prod_{k=1}^n (1 - |\varphi_k(z)|^2) \right]$$

tends to 0 as z approaches the full boundary of \mathbb{D}^n . This settles an outstanding problem concerning composition operators on the polydisk.

1. INTRODUCTION

Let \mathbb{C} be the complex plane and let \mathbb{C}^n be the n -dimensional complex Euclidean space. We use

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

to denote the open unit disk in \mathbb{C} , and use

$$\mathbb{D}^n = \mathbb{D} \times \dots \times \mathbb{D} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_k| < 1, 1 \leq k \leq n\}$$

to denote the polydisk in \mathbb{C}^n . Although we use the same symbol z to denote a point in \mathbb{C} or a point in \mathbb{C}^n , the reader should have no trouble recognizing the meaning of z at every occurrence.

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Let $H(\mathbb{D}^n)$ denote the space of all holomorphic functions in \mathbb{D}^n . For any $0 < p < \infty$ the Hardy space $H^p(\mathbb{D}^n)$ consists of all $f \in H(\mathbb{D}^n)$ such that

$$\|f\|_p^p = \sup_{0 < r < 1} \int_{\mathbb{T}^n} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where

$$\mathbb{T}^n = \{\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : |\zeta_k| = 1, 1 \leq k \leq n\}$$

is the distinguished boundary of \mathbb{D}^n and

$$d\sigma(\zeta) = \frac{|d\zeta_1| \cdots |d\zeta_n|}{(2\pi)^n}$$

is the normalized Haar measure on \mathbb{T}^n . It is well known that for every function $f \in H^p(\mathbb{D}^n)$, the radial limit

$$f(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$$

exists for almost every $\zeta \in \mathbb{T}^n$. Furthermore,

$$\|f\|_p^p = \int_{\mathbb{T}^n} |f(\zeta)|^p d\sigma(\zeta).$$

See [16] for basic information about Hardy spaces of the polydisk.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space $A_\alpha^p(\mathbb{D}^n)$ consists of all functions $f \in H(\mathbb{D}^n)$ such that

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}^n} |f(z)|^p dv_\alpha(z) < \infty,$$

where

$$dv_\alpha(z_1, \dots, z_n) = dA_\alpha(z_1) \cdots dA_\alpha(z_n).$$

Here

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$$

is a weighted area measure on \mathbb{D} with

$$dA(z) = \frac{dx dy}{\pi}$$

being normalized area measure on \mathbb{D} .

We say that a sequence $\{f_k\}$ in A_α^p or H^p converges to 0 weakly if the sequence is bounded in norm and it converges to 0 uniformly on compact sets. We say that a bounded linear operator T from A_α^p or H^p into some $L^p(X, d\mu)$ is compact if

$$\lim_{k \rightarrow \infty} \int_X |Tf_k|^p d\mu = 0$$

whenever $\{f_k\}$ converges to 0 weakly in A_α^p or H^p .

Throughout the paper we fix two positive integers m and n . For any holomorphic mapping $\varphi : \mathbb{D}^m \rightarrow \mathbb{D}^n$, say

$$\varphi(z_1, \dots, z_m) = (\varphi_1(z_1, \dots, z_m), \dots, \varphi_n(z_1, \dots, z_m)),$$

we consider the composition operator

$$C_\varphi : H(\mathbb{D}^n) \rightarrow H(\mathbb{D}^m)$$

defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}^n).$$

The main results of our paper are Theorems A, B, and C below, together with their counterparts when φ maps the polydisk \mathbb{D}^m into the open unit ball \mathbb{B}_n in \mathbb{C}^n .

Theorem A. *For any $p > 0$ and $\alpha > -1$ the composition operator C_φ maps $A_\alpha^p(\mathbb{D}^n)$ boundedly into $A_\beta^p(\mathbb{D}^m)$, where $\beta = n(2 + \alpha) - 2$. Furthermore, the operator*

$$C_\varphi : A_\alpha^p(\mathbb{D}^n) \rightarrow A_\beta^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{z \rightarrow \partial \mathbb{D}^m} \prod_{k=1}^n \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi_k(z_1, \dots, z_m)|^2} = 0,$$

where $\partial \mathbb{D}^m$ is the full topological boundary of \mathbb{D}^m .

We also show that the constant β in Theorem A is the best possible.

Theorem B. *Suppose $p > 0$ and $n > 1$. Then the composition operator C_φ maps $H^p(\mathbb{D}^n)$ boundedly into $A_{n-2}^p(\mathbb{D}^m)$. Furthermore, the operator*

$$C_\varphi : H^p(\mathbb{D}^n) \rightarrow A_{n-2}^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{z \rightarrow \partial \mathbb{D}^m} \prod_{k=1}^n \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi_k(z_1, \dots, z_m)|^2} = 0.$$

Theorem C. *Suppose $p > 0$ and $\varphi : \mathbb{D}^m \rightarrow \mathbb{D}$ is holomorphic, then the composition operator C_φ maps $H^p(\mathbb{D})$ boundedly into $H^p(\mathbb{D}^m)$. Furthermore, the operator*

$$C_\varphi : H^p(\mathbb{D}) \rightarrow H^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{1 - |z|^2} = 0,$$

where N_φ is a certain version of the Nevanlinna counting function of φ .

The case $m = 1$ in Theorems A and B was considered in [21]. Our proofs here are, in fact, reductions to this case.

Composition operators acting on spaces of analytic functions in the unit disk have been well understood. In particular, every analytic self-map φ of the unit disk induces a bounded composition operator on $H^p(\mathbb{D})$ and on $A_\alpha^p(\mathbb{D})$. Furthermore, the operator C_φ is compact on $A_\alpha^p(\mathbb{D})$ if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{1 - |z|^2}{1 - |\varphi(z)|^2} = 0;$$

and the operator C_φ is compact on $H^p(\mathbb{D})$ if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{1 - |z|^2} = 0.$$

The monographs [5], [20] and [23] contain more information about composition operators on the unit disk. One-dimensional results for other spaces of holomorphic functions in the unit disk can be found in [1], [15] [3] and others.

Composition operators acting on spaces of holomorphic functions of several variables have been known to behave in much more complicated ways, see [5],[9],[10],[12],[14],[22]. Although we have determined here the optimal range spaces for the action of C_φ on Hardy and Bergman spaces of \mathbb{D}^n when φ is a holomorphic self-maps of the polydisk \mathbb{D}^n , the corresponding problem for the unit ball is still open. The papers [12], [4], [13], [9] and [10] made partial progress on this problem.

2. PRELIMINARIES ON CARLESON MEASURES

An important tool in the study of composition operators on holomorphic function spaces is the notion of Carleson measures. We will talk about Carleson measures for Hardy spaces as well as Carleson measures for Bergman spaces.

Let Ω denote either the polydisk \mathbb{D}^n or the unit ball \mathbb{B}_n . A positive Borel measure on Ω is called a Carleson measure for the Hardy space $H^p(\Omega)$ if there exists a positive constant $C > 0$ such that

$$\int_{\Omega} |f(z)|^p d\mu(z) \leq C \|f\|_p^p$$

for all $f \in H^p(\Omega)$, where $\|f\|_p$ is the “norm” of f in $H^p(\Omega)$.

We say that μ is a vanishing Carleson measure for the Hardy space $H^p(\Omega)$ if

$$\lim_{k \rightarrow \infty} \int_{\Omega} |f_k(z)|^p d\mu(z) = 0$$

whenever $\{f_k\}$ is a sequence in $H^p(\Omega)$ that converges to 0 weakly.

Lemma 1. *For a positive Borel measure μ on Ω the following two conditions are equivalent.*

- (a) μ is a Carleson measure for $H^p(\Omega)$ for some $p > 0$.
- (b) μ is a Carleson measure for $H^p(\Omega)$ for every $p > 0$.

An analogous characterization also holds for vanishing Carleson measures for $H^p(\Omega)$.

Proof. See [5] and [24] for the case $\Omega = \mathbb{B}_n$. See [2] and [8] for the case $\Omega = \mathbb{D}^n$. \square

Similarly, a positive Borel measure μ on Ω is called a Carleson measure for the Bergman space $A_\alpha^p(\Omega)$ if there exists a constant $C > 0$ such that

$$\int_{\Omega} |f(z)|^p d\mu(z) \leq C \int_{\Omega} |f(z)|^p dv_\alpha(z)$$

for all $f \in A_\alpha^p(\Omega)$. And μ is called a vanishing Carleson measure for $A_\alpha^p(\Omega)$ if

$$\lim_{k \rightarrow \infty} \int_{\Omega} |f_k(z)|^p d\mu(z) = 0$$

whenever $\{f_k\}$ is a sequence in $A_\alpha^p(\Omega)$ that converges to 0 weakly.

Lemma 2. *Suppose $\alpha > -1$ and μ is a positive Borel measure on Ω . Then the following two conditions are equivalent.*

- (a) μ is a Carleson measure for $A_\alpha^p(\Omega)$ for some $p > 0$.
- (b) μ is a Carleson measure for $A_\alpha^p(\Omega)$ for every $p > 0$.

Proof. See [5] and [25]. \square

3. THE ACTION OF C_φ ON BERGMAN SPACES

In this section we study the action of C_φ on Bergman spaces when φ is a holomorphic mapping from \mathbb{D}^m into \mathbb{D}^n .

Theorem 3. *For any $p > 0$ and $\alpha > -1$ the operator C_φ maps $A_\alpha^p(\mathbb{D}^n)$ boundedly into $A_\beta^p(\mathbb{D}^m)$, where $\beta = n(2 + \alpha) - 2$.*

Proof. It is easy to check that biholomorphic maps of \mathbb{D}^n induce bounded composition operators on Bergman and Hardy spaces of \mathbb{D}^n . Therefore, by an obvious change of variables if necessary, we may assume that $\varphi(0) = 0$. We also assume that $m > 1$; the case $m = 1$ was considered in [21].

Let S^{2m-1} as usual stands for the unit sphere in \mathbb{C}^m and $\mathbb{C}\mathbb{P}^k$ for the k -dimensional complex projective space. Each $l = [\zeta_1, \dots, \zeta_m] \in \mathbb{C}\mathbb{P}^{m-1}$ with $\sum_{j=1}^m |\zeta_j|^2 = 1$ intersects \mathbb{D}^m by a one-dimensional disk \mathbb{D}_l of radius

$r_l = \frac{1}{\max\{|\zeta_j|:j=1,\dots,m\}}$. We remark that $1 \leq r_l \leq \sqrt{m}$. For $l \in \mathbb{C}\mathbb{P}^{m-1}$ let us denote by $\tau_l(w; \alpha)$ the following function in the unit disk \mathbb{D}

$$\tau_l(w; \alpha) = \prod_{j=1}^m (1 - r_l^2 |\zeta_j w|^2)^\alpha.$$

We obviously have

$$(1) \quad \tau_l(w; \alpha) \leq (1 - |w|^2)^\alpha$$

It is easily seen that the A_α^p -norm on \mathbb{D}^m is equivalent to

$$(2) \quad \left(\int_{\mathbb{C}\mathbb{P}^{m-1}} \left(\int_{\mathbb{D}} |f(r_l w \zeta)|^p \tau_l(w; \alpha) dA(w) \right) \omega^{m-1}(l) \right)^{1/p} \\ = \left(\int_{S^{2m-1}} \left(\int_{\mathbb{D}} |f(r_l w \zeta)|^p \tau_{[\zeta]}(w; \alpha) dA(w) \right) d\sigma(\zeta) \right)^{1/p},$$

where ω^{m-1} is the Fubini-Study volume form on $\mathbb{C}\mathbb{P}^{m-1}$, $d\sigma$ is the measure on S^{2m-1} invariant under the action of the unitary group and $l = [\zeta_1, \dots, \zeta_m]$.

For $\zeta \in S^{2m-1}$ we denote by $\psi_\zeta : \mathbb{D} \rightarrow \mathbb{D}^m$ the mapping given by

$$\psi_\zeta(w) = r_l w \zeta,$$

where as above $r_l = \frac{1}{\max\{|\zeta_j|:j=1,\dots,m\}}$. Then $g_\zeta = \varphi \circ \psi_\zeta$ maps the unit disk \mathbb{D} into \mathbb{D}^n and sends the origin in \mathbb{C} to the origin in \mathbb{C}^n . By Theorem 4 of [21], there exists a constant $C > 0$, independent of ζ , such that for every function $f \in A_\alpha^p(\mathbb{D}^n)$ we have

$$\int_{\mathbb{D}} |f(g_\zeta(w))|^p dA_\beta(w) \leq C \int_{\mathbb{D}^n} |f(z)|^p dv_\alpha(z).$$

Since $d\sigma$ is a probability measure, the result follows from (1) and (2). \square

Theorem 4. *Suppose $p > 0$, $\alpha > -1$, and $\beta = n(2 + \alpha) - 2$. Then the operator*

$$C_\varphi : A_\alpha^p(\mathbb{D}^n) \rightarrow A_\beta^p(\mathbb{D}^m)$$

is compact if and only if

$$(3) \quad \lim_{z \rightarrow \partial \mathbb{D}^m} \prod_{k=1}^n \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi_k(z_1, \dots, z_m)|^2} = 0.$$

Proof. Fix α , β , and φ . We define the pull-back measure μ in a standard way: for any Borel set E in \mathbb{D}^n

$$\mu(E) = v_\beta(\varphi^{-1}(E)).$$

Then μ is a positive Borel measure on \mathbb{D}^n and it follows from standard measure theory that we have the following change of variables formula:

$$\int_{\mathbb{D}^m} |f(\varphi)|^p dv_\beta = \int_{\mathbb{D}^n} |f|^p d\mu.$$

Therefore, the composition operator

$$C_\varphi : A_\alpha^p(\mathbb{D}^n) \rightarrow A_\beta^p(\mathbb{D}^m)$$

is compact if and only if the measure μ is a vanishing Carleson measure for the Bergman space $A_\alpha^p(\mathbb{D}^n)$. According to Lemma 2, the condition that μ is a vanishing Carleson measure for $A_\alpha^p(\mathbb{D}^n)$ is independent of p . Therefore, we may assume $p = 2$ without loss of generality.

Also, any biholomorphic mapping of \mathbb{D}^n induces an invertible composition operator on $A_\alpha^p(\mathbb{B}_n)$. The same goes for Hardy spaces as well. Therefore, we may assume $\varphi(0) = 0$ without loss of generality.

For the rest of the proof we assume that $p = 2$, $\varphi(0) = 0$, and $m > 1$. The case $m = 1$ was considered in [21].

We can consider the adjoint operator

$$C_\varphi^* : A_\beta^2(\mathbb{D}^m) \rightarrow A_\alpha^2(\mathbb{D}^n).$$

It is easy to see that, for the normalized reproducing kernels of $A_\beta^2(\mathbb{D}^m)$,

$$k_z(w) = \prod_{k=1}^m \frac{(1 - |z_k|^2)^{(2+\beta)/2}}{(1 - w_k \bar{z}_k)^{2+\beta}},$$

we have

$$\int_{\mathbb{D}} |C_\varphi^* k_z|^2 dA_\alpha = \left[\prod_{k=1}^n \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi_k(z_1, \dots, z_m)|^2} \right]^{2+\alpha}.$$

As $z = (z_1, \dots, z_m)$ approaches the full topological boundary $\partial\mathbb{D}^m$ of \mathbb{D}^m , the functions k_z converges to 0 weakly in $A_\beta^2(\mathbb{D}^m)$. If $C_\varphi : A_\alpha^2(\mathbb{D}^n) \rightarrow A_\beta^2(\mathbb{D}^m)$ is compact, then $C_\varphi^* : A_\beta^2(\mathbb{D}^m) \rightarrow A_\alpha^2(\mathbb{D}^n)$ is compact, and so condition (3) must hold.

Conversely, assume we have condition (3). We will use the notation introduced in the proof of Theorem 3. Let us remark that for every $\zeta \in S^{2m-1}$ (3) implies

$$(4) \quad \lim_{|w| \rightarrow 1^-} \prod_{j=1}^n \frac{1 - |w|^2}{1 - |g_{\zeta,j}(w)|^2} = 0,$$

where $g_\zeta = (g_{\zeta,1}, \dots, g_{\zeta,n}) = (\varphi_1(r_l w \zeta), \dots, \varphi_n(r_l w \zeta))$. Indeed, if $|\zeta_j| < \max\{|\zeta_k| : 1 \leq k \leq n\}$ then $1 - |w|^2 < 1 - r_l^2 |w \zeta_j|^2$, and therefore the product in (4) is dominated by the product in (3) computed at $\psi_\zeta(w) = (r_l w \zeta_1, \dots, r_l w \zeta_m)$. Since $\psi_\zeta(w)$ approaches the topological boundary of \mathbb{D}^m as $|w| \rightarrow 1^-$, (4) follows.

Let $\{f_k\}$ be a sequence in $A_\alpha^2(\mathbb{D}^n)$ that converges to 0 weakly and consider the functions

$$F_k(\zeta) = \int_{\mathbb{D}} |f_k(g_\zeta(w))|^2 dA_\beta(w).$$

As it was mentioned in the proof of Theorem 3

$$\begin{aligned} & \|f_k \circ \varphi\|_{2,\beta}^2 = \int_{\mathbb{D}^m} |f_k \circ \varphi|^2 dv_\beta \\ & \sim \int_{S^{2m-1}} \left(\int_{\mathbb{D}} |f_k(g_\zeta(w))|^2 \tau_{[\zeta]}(w; \beta) dA(w) \right) d\sigma(\zeta) \leq \int_{S^{2m-1}} F_k(\zeta) d\sigma(\zeta), \end{aligned}$$

where the last inequality follows from (1).

Since $g_\zeta(0) = 0$ for all $\zeta \in S^{2m-1}$, by Theorem 4 of [21], there exists a constant $C > 0$, independent of k and independent of ζ , such that

$$F_k(\zeta) \leq C.$$

On the other hand, it follows from condition (4) and Theorem 4 of [21] that

$$\lim_{k \rightarrow \infty} F_k(\zeta) = 0.$$

This together with dominated convergence gives

$$\lim_{k \rightarrow \infty} \int_{\mathbb{D}^n} |f_k(\varphi)|^2 dv_\beta = 0.$$

This shows that $C_\varphi : A_\alpha^2(\mathbb{D}^n) \rightarrow A_\beta^2(\mathbb{D}^m)$ is compact whenever condition (3) is satisfied. \square

Remark Our proofs of Theorems 3 and 4 show that similar results hold for holomorphic mappings of an arbitrary bounded symmetric domain in \mathbb{C}^m into \mathbb{D}^n .

We conclude the section with an example which shows that the constant β in Theorem 3 is the best possible. Let $\varphi : \mathbb{D}^m \rightarrow \mathbb{D}^n$ be defined by

$$\varphi(z_1, \dots, z_m) = (z_1, z_1, \dots, z_1).$$

Denote by M the subspace of functions in $A_\alpha^2(\mathbb{D}^n)$ vanishing on the diagonal Δ ,

$$M = \{f \in A_\alpha^2(\mathbb{D}^n) : f(z, z, \dots, z) = 0, z \in \mathbb{D}\}.$$

It follows from [21] that the restriction of functions in $A_\alpha^2(\mathbb{D}^n)$ to the diagonal is a bounded invertible operator from $A_\alpha^2(\mathbb{D}^n) \ominus M$ onto $A_\beta^2(\mathbb{D})$, where

$\beta = n(\alpha + 2) - 2$. Define $\psi : \mathbb{D}^m \rightarrow \mathbb{D}$ by $\psi(z_1, \dots, z_m) = z_1$. It is easy to check that for every $\gamma > -1$ the operator

$$C_\psi : A_\gamma^2(\mathbb{D}) \rightarrow A_\gamma^2(\mathbb{D}^m)$$

is an isometry. If we denote by the same letter Δ the diagonal mapping of the unit disk \mathbb{D} into \mathbb{D}^n , then

$$C_\varphi = C_\psi \circ C_\Delta.$$

This shows that the constant $\beta = n(\alpha + 2) - 2$ is the best possible in Theorem 3.

4. THE ACTION OF C_φ ON HARDY SPACES

In this section we study the action of C_φ on Hardy spaces when φ is a holomorphic map from \mathbb{D}^m into \mathbb{D}^n .

Theorem 5. *Suppose $p > 0$ and φ is a holomorphic map from \mathbb{D}^m into \mathbb{D}^n , where $m \geq 1$ and $n > 1$. Then the composition operator maps $H^p(\mathbb{D}^n)$ boundedly into $A_{n-2}^p(\mathbb{D}^m)$. Furthermore, the operator*

$$C_\varphi : H^p(\mathbb{D}^n) \rightarrow A_{n-2}^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{z \rightarrow \partial \mathbb{D}^m} \prod_{k=1}^n \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{(1 - |\varphi_k(z_1, \dots, z_m)|^2)} = 0.$$

Proof. The proof is similar to that of Theorems 3 and 4. We leave the details to the interested reader. \square

When $n = 1$, it is clear that the space $A_{n-2}^p(\mathbb{D}^m)$ must be replaced by the Hardy space $H^p(\mathbb{D}^m)$. To characterize compactness, we need to introduce a version of the Nevanlinna counting function for holomorphic functions on \mathbb{D}^m .

Recall that if f is an analytic function on the unit disk, the Nevanlinna counting function N_f is defined by

$$(5) \quad N_f(z) = \sum \left\{ \log \frac{1}{|w|} : f(w) = z \right\}, \quad z \neq f(0).$$

The well-known Littlewood's inequality states that if f maps \mathbb{D} into \mathbb{D} and $z \neq f(0)$, then

$$N_f(z) \leq \log \left| \frac{1 - \overline{f(0)}z}{z - f(0)} \right|;$$

see [20], page 187, or [5], page 33 for details. In particular, if $f(0) = 0$, then

$$N_f(z) \leq \log \frac{1}{|z|}, \quad z \neq 0.$$

If f is a holomorphic function from \mathbb{D}^m into \mathbb{D} , where $m > 1$, we consider “slice functions” of the form

$$f_\zeta(w) = f(w\zeta), \quad w \in D,$$

where

$$\zeta = (1, \zeta_2, \dots, \zeta_m) \in \mathbb{T}^m.$$

Each f_ζ is then an analytic self-map of the unit disk. We now define the Nevanlinna counting function of f as follows.

$$(6) \quad N_f(z) = \int_{\mathbb{T}^{m-1}} N_{f_\zeta}(z) d\sigma(\zeta_2, \dots, \zeta_m), \quad z \in \mathbb{D}.$$

It follows directly from the one-dimensional Littlewood inequality that

$$(7) \quad N_f(z) \leq \log \left| \frac{1 - \overline{f(0)}z}{z - f(0)} \right|, \quad z \in \mathbb{D} - \{f(0)\}.$$

We can now settle the case $n = 1$.

Theorem 6. *Suppose $p > 0$ and $\varphi : \mathbb{D}^m \rightarrow \mathbb{D}$ is holomorphic. Then the composition operator C_φ maps $H^p(\mathbb{D})$ boundedly into $H^p(\mathbb{D}^m)$. Moreover, the operator*

$$C_\varphi : H^p(\mathbb{D}) \rightarrow H^p(\mathbb{D}^m)$$

is compact if and only if

$$(8) \quad \lim_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{1 - |z|^2} = 0.$$

Proof. Once again, we may assume that $p = 2$ and $\varphi(0) = 0$.

For any $f \in H^2(\mathbb{D})$ we first use Fubini’s theorem to write

$$\int_{\mathbb{T}^m} |f(\varphi) - f(0)|^2 d\sigma = \int_{\mathbb{T}^{m-1}} d\sigma(\zeta) \int_{\mathbb{T}} |f(\varphi_\zeta(\eta) - f(0))|^2 \frac{|d\eta|}{2\pi}.$$

We then apply the Littlewood-Paley formula (see [6], page 236) to the inner integral above. The result is

$$\int_{\mathbb{T}^m} |f(\varphi) - f(0)|^2 d\sigma = \int_{\mathbb{T}^{m-1}} d\sigma(\zeta) \int_{\mathbb{D}} |f'(\varphi_\zeta(w))|^2 |\varphi'_\zeta(w)|^2 \log \frac{1}{|w|} dA(w).$$

Applying a well-known change of variables formula (see Proposition 10.2.5 of [23] for example) to the inner integral above, we obtain

$$\int_{\mathbb{T}^m} |f(\varphi) - f(0)|^2 d\sigma = \int_{\mathbb{T}^{m-1}} d\sigma(\zeta) \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi_\zeta}(w) dA(w).$$

By Fubini's theorem again, we have

$$(9) \quad \int_{\mathbb{T}^m} |f(\varphi) - f(0)|^2 d\sigma = \int_{\mathbb{D}} |f'(w)|^2 N_\varphi(w) dA(w).$$

It follows from (7) and the Littlewood-Paley identity again that

$$\|f(\varphi) - f(0)\|_{H^2(\mathbb{D}^m)} \leq \|f - f(0)\|_{H^2(\mathbb{D})}$$

for all $f \in H^2(\mathbb{D})$. This proves that the composition operator C_φ is a contraction from $H^2(\mathbb{D})$ into $H^2(\mathbb{D}^m)$.

By equation (9) and the Littlewood-Paley identity, the operator

$$C_\varphi : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^m)$$

is compact if and only if the measure

$$N_\varphi(z) dA(z)$$

is a vanishing Carleson measure for the Bergman space

$$H(\mathbb{D}) \cap L^2(\mathbb{D}, \log(1/|z|) dA(z)).$$

Since $\log(1/|z|)$ is comparable to $1 - |z|^2$ as $|z| \rightarrow 1^-$, we conclude that the compactness of the operator $C_\varphi : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D}^m)$ is equivalent to the condition that the measure

$$d\mu(z) = N_\varphi(z) dA(z)$$

is a vanishing Carleson measure for the Bergman space $A_1^2(\mathbb{D})$. It follows from Lemma 1 of [11] and a standard argument (see Exercise 4 on page 124 of [23]) that the above measure μ is a vanishing Carleson measure for $A_1^2(\mathbb{D})$ if and only if the condition in (8) holds. This completes the proof of the theorem. \square

5. THE CASE WHEN φ MAPS \mathbb{D}^m INTO \mathbb{B}_n

In this section we consider the open unit ball in \mathbb{C}^n ,

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

Here $z = (z_1, \dots, z_n)$ and

$$|z| = \sqrt{|z_1|^2 + \dots + |z_n|^2}.$$

For the above z and $w = (w_1, \dots, w_n)$ we also write

$$\langle z, w \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n.$$

Let $H(\mathbb{B}_n)$ denote the space of all holomorphic functions in \mathbb{B}_n .

Throughout this section we fix a holomorphic map φ from the polydisk \mathbb{D}^m into the unit ball \mathbb{B}_n and consider the composition operator

$$C_\varphi : H(\mathbb{B}_n) \rightarrow H(\mathbb{D}^m).$$

More specifically, we are going to study the action of C_φ on Hardy and weighted Bergman spaces of the unit ball \mathbb{B}_n .

Recall that for $p > 0$ the Hardy space $H^p(\mathbb{B}_n)$ consists of all functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_p^p = \sup_{0 < r < 1} \int_{\mathbb{S}^n} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere

$$\mathbb{S}^n = \partial\mathbb{B}_n = \{\zeta \in \mathbb{C}^n : |\zeta| = 1\}.$$

It is well known that for every function $f \in H^p(\mathbb{B}_n)$, the radial limit

$$f(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$$

exists for almost all $\zeta \in \mathbb{S}^n$. Moreover,

$$\|f\|_p^p = \int_{\mathbb{S}^n} |f(\zeta)|^p d\sigma(\zeta).$$

See [17] and [24] for more information about Hardy spaces of the unit ball.

Similarly, for $p > 0$ and $\alpha > -1$ we consider the weighted Bergman space $A_\alpha^p(\mathbb{B}_n)$ consisting of functions $f \in H(\mathbb{B}_n)$ such that

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) < \infty,$$

where

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z).$$

Here dv is Lebesgue measure on \mathbb{B}_n and c_α is normalizing constant such that $v_\alpha(\mathbb{B}_n) = 1$.

Theorem 7. *For any $p > 0$ and $\alpha > -1$ the composition operator C_φ maps $A_\alpha^p(\mathbb{B}_n)$ boundedly into $A_\beta^p(\mathbb{D}^m)$, where $\beta = n + \alpha - 1$. Furthermore, the operator*

$$C_\varphi : A_\alpha^p(\mathbb{B}_n) \rightarrow A_\beta^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{z \rightarrow \partial\mathbb{D}^m} \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi(z_1, \dots, z_m)|^2} = 0.$$

Proof. This is similar to the proof of Theorems 3 and 4. □

Theorem 8. *Suppose $p > 0$ and $n > 1$. Then the composition operator maps $H^p(\mathbb{B}_n)$ boundedly into $A_\beta^p(\mathbb{D}^m)$, where $\beta = n - 2$. Moreover, the operator*

$$C_\varphi : H^p(\mathbb{B}_n) \rightarrow A_\beta^p(\mathbb{D}^m)$$

is compact if and only if

$$\lim_{z \rightarrow \partial \mathbb{D}^m} \frac{(1 - |z_1|^2) \cdots (1 - |z_m|^2)}{1 - |\varphi(z_1, \dots, z_m)|^2} = 0.$$

Proof. Once again, this is similar to the proof of Theorems 3 and 4. \square

6. FURTHER RESULTS AND REMARKS

It was shown in [9] that if $\varphi : \mathbb{B}_n \rightarrow \mathbb{D}$ is holomorphic, then the composition operator C_φ maps $H^p(\mathbb{D})$ boundedly into $H^p(\mathbb{B}_n)$, and the range space $H^p(\mathbb{B}_n)$ is best possible. Using the same arguments from the proof of Theorem 6, we can prove the following little oh version.

Theorem 9. *If $p > 0$ and $\varphi : \mathbb{B}_n \rightarrow \mathbb{D}$ is holomorphic, then the operator*

$$C_\varphi : H^p(\mathbb{D}) \rightarrow H^p(\mathbb{B}_n)$$

is compact if and only if

$$\lim_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{1 - |z|^2} = 0.$$

Here the Nevanlinna counting function of $\varphi : \mathbb{B}_n \rightarrow \mathbb{D}$ is defined as

$$(10) \quad N_\varphi(z) = \int_{\mathbb{S}^n} N_{\varphi_\zeta}(z) d\sigma(\zeta), \quad z \in \mathbb{D},$$

where for $\zeta \in \mathbb{S}^n$,

$$\varphi_\zeta(z) = \varphi(z\zeta), \quad z \in \mathbb{D},$$

is the standard slice function induced by φ .

We mention a general definition of the Nevanlinna counting function used in the value distribution theory for functions of several complex variables, and point out the relationship between this and our earlier definitions. Our references for this discussion are [7] and [18].

Let A be a k -dimensional analytic set in \mathbb{C}^n and $0 \notin A$. Define

$$(11) \quad N(A, r) = \int_0^r \frac{dt}{t} \int_{A \cap B_t} \omega^k,$$

where $r > 0$, $B_t = t\mathbb{B}_n$ is a ball of radius t , and ω is the Fubini-Study form, namely,

$$\omega = dd^c \log |z|^2.$$

It is well known that

$$(12) \quad N(A, r) = \int_{\tilde{A}} N(A \cap l, r) \omega^k,$$

where \tilde{A} is the projectivization of A and $N(A \cap l, r)$ is the counting function of the divisor $A \cap l$, in the one-dimensional complex line $l \in \mathbb{C}P^{n-1}$ defined in a regular way as the logarithmic average given by (5).

If f is a holomorphic function from \mathbb{C}^n to \mathbb{C} and if $z \neq f(0)$, we define

$$(13) \quad N_f(r, z) = N(A(f, z), r) = \int_{\tilde{A}(f, z)} N(A(f, z) \cap l, r) \omega^{n-1},$$

where

$$A(f, z) = \{w \in \mathbb{B}_n : f(w) = z\}$$

is an $(n - 1)$ -dimensional analytic set in \mathbb{C}^n .

If f is a holomorphic function from \mathbb{B}_n into \mathbb{D} , the Nevanlinna counting function of f can be defined as $N_f(z) = N_f(1, z)$. Since ω^{n-1} is the volume form for the Fubini-Study metric, we can express this version of the counting function as an integral mean of sectional one-dimensional counting functions over the unit sphere. Therefore, this definition of the Nevanlinna counting function coincides with our early definition given in (10) in the case of the unit ball.

In the case of the unit polydisk, the measure used to define the Hardy spaces H^p lives on the torus \mathbb{T}^n which has measure 0 in the full boundary of \mathbb{D}^n . In this case, the above definition based on “projectivization” is not appropriate for the study of composition operators on Hardy spaces, and we must resort to the direct integral form given in (6).

We also mention that our results on compactness of composition operators on Hardy spaces can be strengthened to yield Shapiro’s formula for the essential norm.

Theorem 10. *If φ is a holomorphic function from Ω into \mathbb{D} , where Ω is either the polydisk or the unit ball, then the essential norm of the operator*

$$C_\varphi : H^2(\mathbb{D}) \rightarrow H^2(\Omega)$$

is given by the formula

$$\|C_\varphi\|_e = \limsup_{|z| \rightarrow 1^-} \frac{N_\varphi(z)}{\log \frac{1}{|z|}}.$$

Proof. This follows from our proof of Theorem 6 and Shapiro’s arguments in [19]. \square

Similar to the one-dimensional case, inner functions in either \mathbb{D}^n or \mathbb{B}_n can be described in terms of Nevanlinna counting functions. The following result can be obtained as an adaptation of the corresponding theorem in [19] to the case of functions of several variables.

Theorem 11. *Let Ω be either \mathbb{D}^n or \mathbb{B}_n . A holomorphic mapping $\varphi : \Omega \rightarrow \mathbb{D}$ is inner if and only if*

$$N_\varphi(z) = \log \left| \frac{1 - \overline{\varphi(0)}z}{z - \varphi(0)} \right|$$

for almost all z in $\mathbb{D} \setminus 0$.

Finally, as was pointed out in the introduction, we mention that it is still an open problem to determine the optimal range space for the action of composition operators C_φ on a Hardy or weighted Bergman space when φ is a holomorphic self-map of the unit ball. We consider this a fundamental issue in the theory of composition operators.

REFERENCES

- [1] P. Bourdon, J. Cima and A. Matheson, Compact composition operators on BMOA, *Pacific J. Math.* **351** (1999), 2183-2196.
- [2] S.-Y. Chang, Carleson measure on the bi-disc, *Ann. Math.* **109** (1979), 613-620.
- [3] J. Cima and A. Matheson, Weakly compact composition operators on VMO, *Rocky Mountain J. Math.* **2002**, no. 3, 937-951.
- [4] J. Cima and P. Mercer, Composition operators between Bergman spaces on convex domains, *J. Operator Theory* **33** (1995), 363-369.
- [5] C. Cowen and B. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton, 1994.
- [6] J. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.
- [7] Ph. Griffiths and J. King, Nevanlinna theory and holomorphic mappings between algebraic varieties, *Acta Math.* **130** (1973), 145-220.
- [8] F. Jafari, Carleson measures in Hardy and weighted Bergman spaces of polydiscs, *Proc. Amer. Math. Soc.* **112** (1991), 771-781.
- [9] H. Koo and W. Smith, Composition operators between spaces of functions of several variables, preprint, 2004.
- [10] H. Koo and W. Smith, Composition operators induced by smooth self-mappings of the unit ball in \mathbb{C}^n , preprint, 2005.
- [11] D. Lueking and K. Zhu, Composition operators belonging to the Schatten ideals, *Amer. J. Math.* **114** (1992), 1127-1145.
- [12] B. MacCluer, Compact composition operators on $H^p(\mathbb{B}_n)$, *Michigan J. Math.* **32** (1985), 237-248.
- [13] B. MacCluer and P. Mercer, Composition operators between Hardy and weighted Bergman spaces on convex domains in \mathbb{C}^N , *Proc. Amer. Math. Soc.* **123** (1995), 2093-2102.
- [14] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canadian J. Math.* **38** (1986), 878-906.

- [15] K. Madigan and A. Matheson, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347** (1995), 2679-2687.
- [16] W. Rudin, *Function Theory in Polydiscs*, Benjamin, New York, 1969.
- [17] W. Rudin, *Function Theory in the Unit Ball of \mathbb{C}^n* , Springer, New York, 1980.
- [18] B.V. Shabat, *Distribution of Values of Holomorphic Mappings*, American Mathematical Society, Providence, 1985.
- [19] J. Shapiro, The essential norm of a composition operator, *Ann. Math.* **125** (1987), 375-404.
- [20] J. Shapiro, *Composition operators and classical function theory*, Spriger-Verlag, New York, 1993.
- [21] M. Stessin and K. Zhu, Composition operators on embedded disks, preprint, 2004.
- [22] W.R. Wogen, The smooth mapping which preserve the Hardy space H_B^2 , *Operator Theory: Advances Appl.* **35** (1988), 249-267.
- [23] K. Zhu, *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.
- [24] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, 2004.
- [25] K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory* **20** (1988), 329-357.

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