LACUNARY SERIES IN $\text{Q}_K$ SPACES

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ABSTRACT. Under mild conditions on the weight function $K$ we characterize lacunary series in the so-called $\text{Q}_K$ spaces.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The Green's function for $\mathbb{D}$ is given by

$$g(z, w) = \log \frac{1}{|\sigma_w(z)|} = \log \left| \frac{1 - \overline{w}z}{w - z} \right|,$$

where

$$\sigma_w(z) = \frac{w - z}{1 - \overline{w}z}$$

is a Möbius transformation of $\mathbb{D}$.

Given a function $K : (0, \infty) \to [0, \infty)$, we consider the space $\text{Q}_K$ of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\text{Q}_K}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, w)) \, dA(z) < \infty,$$

where $H(\mathbb{D})$ is the space of all analytic functions in $\mathbb{D}$ and $dA$ is the Euclidean area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$. It is easy to check that $\| \cdot \|_{\text{Q}_K}$ is a complete semi-norm on $\text{Q}_K$ and it is Möbius invariant, that is,

$$\|f \circ \sigma\|_{\text{Q}_K} = \|f\|_{\text{Q}_K}, \quad \sigma \in \text{Aut}(\mathbb{D}),$$

where $\text{Aut}(\mathbb{D})$ is the group of all Möbius maps of the unit disk.

It is clear that each $\text{Q}_K$ contains all constant functions. If $\text{Q}_K$ consists of just the constant functions, we say that it is trivial. It follows from the general theory of Möbius invariant function spaces (see [1] for example) that $\text{Q}_K$ is nontrivial if and only if it contains the coordinate function $z$, and in this case, $\text{Q}_K$ contains all polynomials.

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From a change of variables we see that the coordinate function $z$ belongs to $Q_K$ if and only if

$$\sup_{w \in \mathbb{D}} \int_D \frac{(1 - |w|^2)^2}{|1 - wz|^4} K \left( \log \frac{1}{|z|} \right) dA(z) < \infty.$$ 

Simplifying the above integral in polar coordinates, we conclude that $Q_K$ is nontrivial if and only if

$$\sup_{t \in (0, 1)} \int_0^1 \frac{(1 - t)^2}{(1 - tr^2)^3} K \left( \log \frac{1}{r} \right) r dr < \infty. \quad (1)$$

Throughout the paper we always assume that condition (1) above is satisfied, so that the space $Q_K$ we study is nontrivial. Another standing assumption we make for the rest of the paper is that the weight function $K$ be nondecreasing.

An important tool in the study of $Q_K$ spaces is the auxiliary function $\varphi_K$ defined by

$$\varphi_K(s) = \sup_{0 < t \leq 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$ 

The following condition has played a crucial role in the study of $Q_K$ spaces during the last few years:

$$\int_1^\infty \varphi_K(s) \frac{ds}{s^2} < \infty. \quad (2)$$

See [9][17][18] for example. This condition will be crucial for us here as well. The main result of the paper is the following.

**Theorem.** If $K$ satisfies condition (2), then a lacunary series

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k}$$

belongs to $Q_K$ if and only if

$$\sum_{k=1}^\infty n_k |a_k|^2 K \left( \frac{1}{n_k} \right) < \infty.$$ 

Recall that a function

$$f(z) = \sum_{k=1}^\infty a_k z^{n_k}$$

is called a lacunary series if

$$\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.$$
Such series are often used to construct examples of analytic functions in various function spaces.

A special case is worth mentioning. When \( K(t) = t^p, \) \( 0 \leq p < \infty, \) the resulting \( Q_K \) space is usually denoted by \( Q_p. \) It is well known that \( Q_p \) coincides with BMOA if \( p = 1, \) and \( Q_p \) is the Bloch space \( B \) if \( p > 1. \) We remind the reader that \( B \) consists of analytic functions \( f \) in \( D \) such that
\[
\sup_{z \in D} (1 - |z|^2) |f'(z)| < \infty.
\]
The most interesting case is when \( 0 < p < 1; \) such \( Q_p \) spaces are distinct Möbius invariant Banach spaces that are strictly contained in BMOA. See [19] for the relatively new theory of \( Q_p \) spaces.

It is well known that a lacunary series belongs to BMOA if and only if it is in the Hardy space \( H^2; \) see [5] for example. It is also well known that a lacunary series is in the Bloch space if and only if its Taylor coefficients are bounded; see [20] for example. Lacunary series in \( Q_p \) are characterized in [4]. More specifically, if \( 0 \leq p \leq 1, \) then a lacunary series
\[
f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}
\]
is in \( Q_p \) if and only if
\[
\sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty.
\]
Since the function \( K(t) = t^p \) satisfies condition (2) if and only if \( p < 1, \) our main result covers \( Q_p \) spaces for \( 0 \leq p < 1, \) but it misses the classical case of BMOA (corresponding to \( p = 1). \) Nevertheless, it should be clear from these remarks that condition (2) is very sharp.

2. PRELIMINARIES ON WEIGHT FUNCTIONS

The function theory of \( Q_K \) obviously depends on the properties of \( K. \) Given two weight functions \( K_1 \) and \( K_2, \) we are going to write \( K_1 \lesssim K_2 \) if there exists a constant \( C > 0, \) independent of \( t, \) such that \( K_1(t) \leq C K_2(t) \) for all \( t. \) The notation \( K_1 \gtrsim K_2 \) is used in a similar fashion. When \( K_1 \lesssim K_2 \lesssim K_1, \) we write \( K_1 \approx K_2. \)

It is clear that \( K_1 \lesssim K_2 \) implies \( Q_{K_2} \subset Q_{K_1}. \) In particular, \( K_1 \) and \( K_2 \) give rise to the same \( Q_K \) space whenever \( K_1 \approx K_2. \) The converse is false in general, as is demonstrated by the fact that \( Q_p \) equals the Bloch space for all \( p > 1. \)

In this section we prove several results about the weight function that are needed for subsequent sections and are of some independent interest.
Lemma 1. If
\[ K_1(t) = \begin{cases} K(t), & 0 < t \leq 1 \\ K(1), & 1 \leq t < \infty \end{cases} \]
then \( Q_K = Q_{K_1} \).

Proof. Since \( K \) is nondecreasing, we have \( K_1 \leq K \), so \( Q_K \subseteq Q_{K_1} \). In particular, both spaces are nontrivial Möbius invariant spaces.

Since \( K(\log \frac{1}{|z|}) \) is a radial function, integration in polar coordinates shows that \( f \mapsto f'(0) \) is a bounded linear functional on any nontrivial \( Q_K \) space. By [12], each such space \( Q_K \) is contained in the Bloch space.

Fix a function \( f \in Q_{K_1} \) and consider the integrals
\[ I(a) = \int_D |f'(z)|^2 K(g(z, a)) \, dA(z). \]
We must show that \( I(a) \) is bounded for \( a \in \mathbb{D} \). To this end, we write \( I(a) = I_1(a) + I_2(a) \), where
\[ I_1(a) = \int_{|\varphi_a(z)| > e^{-1}} |f'(z)|^2 K(g(z, a)) \, dA(z), \]
and
\[ I_2(a) = \int_{|\varphi_a(z)| \leq e^{-1}} |f'(z)|^2 K(g(z, a)) \, dA(z). \]

It is clear that
\[ I_1(a) \leq \int_D |f'(z)|^2 K_1(g(z, a)) \, dA(z), \]
so there exists a positive constant \( C_1 \) such that \( I_1(a) \leq C_1 \) for all \( a \in \mathbb{D} \).

By a change of variables, we have
\[ I_2(a) = \int_{|\varphi_a(z)| \leq e^{-1}} |f'(z)|^2 K \left( \frac{1}{|\varphi_a(z)|} \right) \, dA(z) \]
\[ = \int_{|z| \leq e^{-1}} |f'(\varphi_a(z))|^2 K \left( \frac{1}{|z|} \right) \frac{(1 - |a|^2)^2}{|1 - az|^4} \, dA(z) \]
\[ = \int_{|z| \leq e^{-1}} |f'(\varphi_a(z))|^2 \left( 1 - |\varphi_a(z)|^2 \right)^2 \frac{K \left( \log \frac{1}{|z|} \right)}{(1 - |z|^2)^2} \, dA(z). \]

Since \( f \) is in the Bloch space, we can find a constant \( C_2 > 0 \) such that
\[ I_2(a) \leq C_2 \int_{|z| \leq e^{-1}} K \left( \log \frac{1}{|z|} \right) \, dA(z) \leq C_2 \int_D K \left( \log \frac{1}{|z|} \right) \, dA(z). \]
By condition (1), the last integral above is convergent, so there exists a constant $C_3 > 0$ such that $I_2(a) \leq C_3$ for all $a \in \mathbb{D}$. This shows that $I(a)$ is bounded in $a$, or equivalently, $f$ belongs to $Q_K$. □

The significance of Lemma 1 is that the space $Q_K$ only depends on the behavior of $K(t)$ for $t$ close to 0. In particular, when studying $Q_K$ spaces, we can always assume that $K(t) = K(1)$ for $t \geq 1$. However, we do not make this assumption in our main theorems.

**Lemma 2.** If $K$ satisfies condition (2), then the function

$$K^*(t) = t \int_t^\infty K(s) \frac{ds}{s^2}, \quad 0 < t < \infty,$$

has the following properties:

(i) $K^*$ is nondecreasing on $(0, \infty)$.

(ii) $K^*/t$ is nonincreasing on $(0, \infty)$.

(iii) $K^*(t) \geq K(t)$ for all $t \in (0, \infty)$.

(iv) $K^* \lesssim K$ on $(0, 1]$.

If $K(t) = K(1)$ for $t \geq 1$, then we also have

(v) $K^*(t) = K^*(1) = K(1)$ for $t \geq 1$, so $K^* \approx K$ on $(0, \infty)$.

**Proof.** If $t \in (0, 1]$, then a change of variables gives

$$K^*(t) = t \int_t^\infty K(s) \frac{ds}{s^2} = \int_1^\infty K(ts) \frac{ds}{s^2}$$

$$= K(t) \int_1^\infty \frac{K(ts)}{K(t)} \frac{ds}{s^2} \leq K(t) \int_1^\infty \varphi_K(s) \frac{ds}{s^2}.$$

So condition (2) implies that $K^*(t) \lesssim K(t)$ for $t \in (0, 1]$. This yields property (iv) and shows that $K^*(t)$ is well defined for all $t > 0$.

Since

$$\frac{K^*(t)}{t} = \int_t^\infty K(s) \frac{ds}{s^2}$$

and $K$ is nonnegative, we see that the function $K^*/t$ is decreasing. This proves (ii). Property (v) follows from a direct calculation.

Using the assumption that $K$ is nondecreasing again, we obtain

$$K^*(t) = t \int_t^\infty K(s) \frac{ds}{s^2} \geq tK(t) \int_t^\infty \frac{ds}{s^2} = K(t)$$

for all $0 < t < \infty$. This proves property (iii).
It remains for us to show that $K^*$ is nondecreasing. To this end, we fix $0 < t < T < \infty$ and consider the difference

$$D = K^*(T) - K^*(t) = T \int_T^\infty \frac{K(s)}{s^2} \, ds - t \int_t^\infty \frac{K(s)}{s^2} \, ds$$

$$= (T - t) \int_T^\infty \frac{K(s)}{s^2} \, ds - t \int_t^T \frac{K(s)}{s^2} \, ds.$$  

Since $K$ is nondecreasing and nonnegative, we have

$$D \geq (T - t) K(T) \int_T^\infty \frac{ds}{s^2} - t K(T) \int_t^T \frac{ds}{s^2} = 0.$$  

This proves property (i) and completes the proof of the lemma. 

Note that condition (2) is critically needed only in the proof of (iv). Without condition (2), properties (i), (ii), and (iii) remain valid, provided that $K^*$ is allowed to be identically infinite.

**Corollary 3.** If $K$ satisfies condition (2), then there exists a constant $C > 0$ such that $K(2t) \leq CK(t)$ for all $0 \leq 2t \leq 1$.

**Proof.** For any $t > 0$, we have

$$\frac{K^*(2t)}{K^*(t)} = 2 \frac{\int_{2t}^\infty K(s) \, ds}{s^2} \leq 2.$$

The desired estimate now follows from parts (iii) and (iv) of Lemma 2. 

If we started out with a weight function $K$ with the property that $K(t) = K(1)$ for $t \geq 1$, then the conclusion of Corollary 3 can be strengthened to be $K'(2t) \approx K'(t)$ for $t > 0$.

**Proposition 4.** If $K$ satisfies condition (2), then we can find another nonnegative weight function $K^*$ such that $Q_K = Q_{K^*}$ and that the new weight function $K^*$ has the following properties:

(a) $K^*$ is nondecreasing on $(0, \infty)$.
(b) $K^*$ satisfies condition (1).
(c) $K^*$ satisfies condition (2).
(d) $K^*(2t) \approx K^*(t)$ on $(0, \infty)$.
(e) $K^*$ is differentiable (up to any given order) on $(0, \infty)$.
(f) $K^*$ is concave on $(0, \infty)$.
(g) $K^*(t) = K^*(1)$ for $t \geq 1$.
(h) $K^*(t)/t$ is nonincreasing on $(0, \infty)$.
(i) $K^*(t) \approx K(t)$ on $(0, 1]$. 


Proof. By Lemma 1, we may assume that \( K(t) = K(1) \) for all \( t \geq 1 \). Under this assumption, the function \( K^* \) from Lemma 2 then satisfies \( K^* \approx K \) on \((0, \infty)\). Moreover, properties (a), (b), (c), (g), (h), and (i) all hold.

Property (d) follows from the proof of Corollary 3.

If we repeat the construction \( K \mapsto K^* \), then we can make the new weight function differentiable up to any desired order. So property (e) holds.

If the function \( K \) is differentiable, which we may assume by property (e), then

\[
\frac{d}{dt}K^*(t) = \int_t^\infty \frac{K(s) \, ds}{s^2} - \frac{K(t)}{t},
\]

and

\[
\frac{d^2}{dt^2}K^*(t) = -\frac{K'(t)}{t} \leq 0.
\]

This shows that \( K^* \) is concave on \((0, \infty)\) and completes the proof of the proposition. \( \square \)

Theorem 5. If \( K \) satisfies condition (2), then for any \( \alpha \geq 1 \) and \( 0 \leq \beta < 1 \) we have

\[
\int_0^1 r^{\alpha-1} \left( \log \frac{1}{r} \right)^{-\beta} K \left( \log \frac{1}{r} \right) \, dr \approx C(\beta) \left( \frac{1 - \beta}{\alpha} \right)^{1-\beta} K \left( \frac{1 - \beta}{\alpha} \right),
\]

where \( C(\beta) \) is a constant depending on \( \beta \) alone.

Proof. Let

\[
I = \int_0^1 r^{\alpha-1} \left( \log \frac{1}{r} \right)^{-\beta} K \left( \log \frac{1}{r} \right) \, dr.
\]

By a change of variables,

\[
I = \int_0^\infty e^{-\alpha t} t^{-\beta} K(t) \, dt.
\]

We write \( I = I_1 + I_2 \), where

\[
I_1 = \int_0^{1/\alpha} e^{-\alpha t} t^{-\beta} K(t) \, dt,
\]

and

\[
I_2 = \int_{1/\alpha}^{\infty} e^{-\alpha t} t^{-\beta} K(t) \, dt.
\]

Since \( K \) is nondecreasing, we have

\[
I_1 \leq K \left( \frac{1 - \beta}{\alpha} \right) \int_0^{1/\alpha} e^{-\alpha t} t^{-\beta} \, dt.
\]
Making the change of variables $t = (1 - \beta) s / \alpha$, we obtain

$$I_1 \leq \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K \left( \frac{1 - \beta}{\alpha} \right) \int_0^1 e^{-t (1 - \beta) s} s^{-\beta} ds$$

$$= C(\beta) \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K \left( \frac{1 - \beta}{\alpha} \right).$$

By part (iii) of Lemma 2, we have

$$I_2 \leq \int_1^\infty e^{-at} t^{1 - \beta} K^*(t) dt.$$

According to part (ii) of Lemma 2, the function $K^*(t)/t$ is decreasing on $(0, \infty)$, so

$$I_2 \leq K^* \left( \frac{1 - \beta}{\alpha} \right) \frac{1 - \beta}{\alpha} \int_1^\infty e^{-at} t^{1 - \beta} dt.$$

A change of variables ($t = (1 - \beta) s / \alpha$) in the integral above leads to

$$I_2 \leq \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K^* \left( \frac{1 - \beta}{\alpha} \right) \int_1^\infty e^{-t (1 - \beta) s} s^{1 - \beta} ds.$$

This together with part (iv) of Lemma 2 shows that

$$I_2 \lesssim C(\beta) \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K \left( \frac{1 - \beta}{\alpha} \right).$$

Combining this with what was proved in the previous paragraph, we have

$$I \lesssim C(\beta) \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K \left( \frac{1 - \beta}{\alpha} \right).$$

On the other hand, we have

$$I \geq \int_1^\infty e^{-at} t^{1 - \beta} K(t) dt.$$

The assumption that $K$ is nondecreasing gives

$$I \geq K \left( \frac{1 - \beta}{\alpha} \right) \int_1^\infty e^{-at} t^{1 - \beta} dt.$$

Make a change of variables according to $t = (1 - \beta) s / \alpha$. Then

$$I \geq C(\beta) \left( \frac{1 - \beta}{\alpha} \right)^{1 - \beta} K \left( \frac{1 - \beta}{\alpha} \right).$$

This completes the proof of the theorem. □
3. Lacunary Series in $Q_K$

We begin with an estimate of the weighted Dirichlet integral in terms of Taylor coefficients.

**Theorem 6.** If $K$ satisfies condition (2) and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

then

$$\int_{D} |f'(z)|^2 K \left( \log \frac{1}{|z|} \right) \, dA(z) \approx \sum_{n=1}^{\infty} n |a_n|^2 K \left( \frac{1}{n} \right).$$

**Proof.** Write

$$I(f) = \int_{D} |f'(z)|^2 K \left( \log \frac{1}{|z|} \right) \, dA(z).$$

Integrating in polar coordinates leads to

$$I(f) = 2 \sum_{n=1}^{\infty} n^2 |a_n|^2 \int_{0}^{1} r^{2n-1} K \left( \log \frac{1}{r} \right) \, dr.$$

We apply Theorem 5 with $\beta = 0$ and $\alpha = 2n$ to obtain

$$I(f) \approx \sum_{n=1}^{\infty} n |a_n|^2 K \left( \frac{1}{2n} \right).$$

The desired result then follows from Corollary 3. \qed

We are now ready to prove the main result of the paper.

**Theorem 7.** If $K$ satisfies condition (2), then a lacunary series

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

belongs to $Q_K$ if and only if

$$\sum_{k=1}^{\infty} n_k |a_k|^2 K \left( \frac{1}{n_k} \right) < \infty.$$  (3)

**Proof.** First assume that

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$
is a lacunary series in $Q_K$. Then

$$\int_{\mathbb{D}} |f'(z)|^2 K \left( \log \frac{1}{|z|} \right) \, dA(z) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, 0)) \, dA(z) < \infty,$$

which, according to Theorem 6, implies condition (3).

Next assume that condition (3) holds. We proceed to estimate the integral

$$I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z), \quad a \in \mathbb{D}.$$

As the first step, we show that for any $a \in \mathbb{D},$

$$I(a) \leq 2 \int_0^1 r \left[ \sum_{n=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \, dr. \quad (4)$$

To this end, we write $z = re^{i\theta}$ in polar form and observe that

$$|f'(z)| \leq \sum_{n=1}^{\infty} n_k |a_k| r^{n_k-1}.$$

It follows that

$$I(a) \leq 2 \int_0^1 r \left[ \sum_{n=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 r \, dr \int_0^{2\pi} K(g(re^{i\theta}, a)) \, d\theta.$$

By Proposition 4, we may as well assume that $K$ is concave. Then

$$\frac{1}{2\pi} \int_0^{2\pi} K(g(re^{i\theta}, a)) d\theta \leq K \left( \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta \right).$$

By Jensen’s formula, the integral

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1 - \bar{a}re^{i\theta}}{re^{i\theta} - a} \, d\theta$$

is equal to $\log \frac{1}{|a|}$ for $0 < r \leq |a|$ and $\log \frac{1}{r}$ for $|a| < r < 1$. In particular,

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) d\theta \leq \log \frac{1}{r}.$$

From this we deduce inequality (4).

Our second step is to prove that inequality (4) implies

$$I(a) \lesssim \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2 \frac{1}{2^n} K \left( \frac{1}{2^n} \right), \quad (5)$$

where

$$I_n = \{ k : 2^n \leq k < 2^{n+1}, k \in \mathbb{N} \}.$$
To this end, we combine the elementary estimates
\[
\sum_{n=0}^{\infty} 2^{\frac{n}{2}} r^{2n} \leq \sqrt{2} \sum_{n=0}^{\infty} \int_{2^n}^{2^{n+1}} t^{-\frac{1}{2}} r^2 \, dt \\
\leq \sqrt{2} \int_{0}^{\infty} t^{-\frac{1}{2}} r^2 \, dt \\
= 2 \Gamma \left( \frac{1}{2} \right) \left( \log \frac{1}{r} \right)^{-\frac{1}{2}}
\]
with the Cauchy-Schwarz inequality to produce
\[
\left[ \sum_{k=1}^{\infty} n_k |a_k|^r n_k \right]^2 = \left[ \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k|^r n_k \right]^2 \leq \left[ \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k|^r n_k \right]^2 \\
\leq \left[ \sum_{n=0}^{\infty} 2^{\frac{n}{2}} r^{2n} \right] \left[ \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} r^{2n} \left( \sum_{n_k \in I_n} n_k |a_k| \right)^2 \right] \\
\leq \frac{2 \Gamma \left( \frac{1}{2} \right)}{(\log \frac{1}{r})^\frac{1}{2}} \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} r^{2n} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2.
\]
This together with (4) and Theorem 5 and Corollary 3 gives
\[
I(a) \leq 2 \int_{0}^{1} r^{-1} \left[ \sum_{k=1}^{\infty} n_k |a_k|^r n_k \right]^2 K \left( \log \frac{1}{r} \right) \, dr \\
\leq \sum_{n=0}^{\infty} 2^{-\frac{n}{2}} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2 \int_{0}^{1} r^{2n-1} \left( \log \frac{1}{r} \right)^{-\frac{1}{2}} K \left( \log \frac{1}{r} \right) \, dr \\
\leq \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \right]^2 \frac{1}{2^n} K \left( \frac{1}{2^n} \right).
\]
This shows that inequality (5) holds.

If \( n_k \in I_n \), then \( n_k < 2^{n+1} \). It follows from the monotonicity of \( K \) and Corollary 3 that
\[
\frac{1}{n_k} K \left( \frac{1}{n_k} \right) \geq \frac{1}{2^{n+1}} K \left( \frac{1}{2^{n+1}} \right) \geq \frac{1}{2^n} K \left( \frac{1}{2^n} \right).
\]
Combining this with (5), we obtain

$$I(a) \lesssim \sum_{n=0}^{\infty} \left[ \sum_{n_k \in I_n} n_k |a_k| \sqrt{\frac{1}{n_k} K \left( \frac{1}{n_k} \right)} \right]^2.$$  \hspace{1cm} (6)

Note that everything so far in the proof works for an arbitrary analytic function, not just for a lacunary series.

Our final step makes use of the fact that $f$ is a lacunary series. More specifically, if

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1$$

for all $k$, the Taylor series of $f(z)$ has at most $[\log_\lambda 2] + 1$ terms $a_k z^{n_k}$ such that $n_k \in I_n$ for $n \in \mathbb{N}$. By (6) and Hölder’s inequality,

$$I(a) \lesssim ([\log_\lambda 2] + 1) \sum_{n=0}^{\infty} \sum_{n_k \in I_n} n_k |a_k|^2 K \left( \frac{1}{n_k} \right)$$

$$= ([\log_\lambda 2] + 1) \sum_{k=1}^{\infty} n_k |a_k|^2 K \left( \frac{1}{n_k} \right).$$

This shows that condition (3) implies $f \in Q_K$. The proof of the theorem is now complete. \hfill \Box

4. Lacunary Series in $Q_{K,0}$

Let $Q_{K,0}$ denote the subspace of $Q_K$ consisting of functions $f$ with

$$\lim_{|a| \to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z) = 0.$$  

The following result together with Theorem 7 characterizes lacunary series in $Q_{K,0}$.

**Theorem 8.** Let

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

be a lacunary series. If $K$ satisfies condition (2), then $f \in Q_K$ if and only if $f \in Q_{K,0}$.

**Proof.** Suppose the lacunary series $f$ belongs to $Q_K$. We must show that $I(a) \to 0$ as $|a| \to 1^-$, where

$$I(a) = \int_{\mathbb{D}} |f'(z)|^2 K(g(z, a)) \, dA(z), \quad a \in \mathbb{D}.$$
From the proof of Theorem 7, we know that $f \in Q_K$ implies that
\[
\int_0^1 r \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \ dr < \infty.
\]
Thus for any given $\varepsilon > 0$ there exists some $\sigma \in (0, 1)$ such that
\[
2 \int_{\sigma}^1 r \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) \ dr < \varepsilon.
\]
We may assume that
\[
\lim_{|a| \to 1^-} K \left( \log \frac{1}{|a|} \right) = 0.
\]
Otherwise, $Q_K$ coincides with the Dirichlet space $D$ (see [8]), and the desired result is obvious.

We write $I(a) = I_1(a) + I_2(a)$, where
\[
I_1(a) = \int_{|z| < \sigma} |f'(z)|^2 K(g(z, a)) \, dA(z)
\]
and
\[
I_2(a) = \int_{\sigma \leq |z| < 1} |f'(z)|^2 K(g(z, a)) \, dA(z).
\]
By arguments used in the second paragraph of the proof of Theorem 7, we have
\[
I_1(a) \leq 2K \left( \log \frac{1}{|a|} \right) \int_{\sigma}^1 r \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 r \ dr
\]
whenever $\sigma < |a| < 1$, because in this case
\[
\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}, a) \, d\theta = \log \frac{1}{|a|}.
\]
In particular, $I_1(a) \to 0$ as $|a| \to 1^-$. Similarly, we have
\[
I_2(a) \leq 2 \int_{\sigma}^1 r \left[ \sum_{k=1}^{\infty} n_k |a_k| r^{n_k-1} \right]^2 K \left( \log \frac{1}{r} \right) r \ dr < \varepsilon.
\]
It follows that
\[
\limsup_{|a| \to 1^-} I(a) \leq \varepsilon.
\]
Since $\varepsilon$ is arbitrary, we conclude that $I(a) \to 0$ as $|a| \to 1^-$. So $f \in Q_{K,0}$ and the proof is complete. \qed
Carefully checking the proof of Theorems 7 and 8, we also obtain the following sufficient condition for a function to be in $Q_{K,0}$ (and hence in $Q_K$) in terms of Taylor coefficients.

**Theorem 9.** If $K$ satisfies condition (2), and if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

satisfies the condition

$$\sum_{n=0}^{\infty} \left[ \sum_{k \in I_n} k |a_k| \right]^2 \frac{1}{2^n} K \left( \frac{1}{2^n} \right) < \infty,$$

then $f \in Q_{K,0}$.

**Proof.** We leave the details to the interested reader. \qed

**REFERENCES**


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