FOCK-SOBOLEV SPACES AND THEIR CARLESON MEASURES

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ABSTRACT. We consider the Fock-Sobolev space $F^{p,m}$ consisting of entire functions f such that $f^{(m)}$, the m-th order derivative of f, is in the Fock space F^p. We show that an entire function f is in $F^{p,m}$ if and only if the function $z^m f(z)$ is in F^p. We also characterize the Carleson measures for the spaces $F^{p,m}$, establish the boundedness of the weighted Fock projection on appropriate L^p spaces, identify the Banach dual of $F^{p,m}$, and compute the complex interpolation space between two $F^{p,m}$ spaces.

1. INTRODUCTION

Let $\mathbb{C}$ be the complex plane and dA be ordinary area measure on $\mathbb{C}$. For any $0 < p \le \infty$ let F^p denote the space of entire functions f such that the function $f(z)e^{-|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. For $f \in F^p$ we write

$$\|f\|_p = \left[\frac{p}{2\pi}\int_{\mathbb{C}} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p dA(z)\right]^{\frac{1}{p}}, \quad 0 < p < \infty,$$

and

$$\|f\|_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{1}{2}|z|^2}.$$

The spaces F^p, especially F^2, have had a long history in mathematics and mathematical physics and have been given a wide variety of appellations, including many combinations and permutations of the names Bargmann, Fischer, Fock, and Segal. See [1, 2, 3, 4, 5, 6, 9, 10, 11, 14]. In this paper we are going to call them Fock spaces, for no particular reason other than personal tradition. We refer the reader to [14, 15] for more recent and systematic treatment of Fock spaces.

To give a motivation for our study of Fock-Sobolev spaces, recall that the annihilation operator A and the creation operator A^* from the quantum theory of harmonic oscillators are defined by the commutation relation $[A, A^*] = I$, where I is the identity operator. A natural representation of these operators is achieved on the Fock space F^2, namely,

$$Af(z) = f'(z), \qquad A^*f(z) = zf(z), \qquad f \in F^2.$$

It is easy to check that both A and A^*, as defined above, are densely defined linear operators on F^2 (unbounded though) and satisfy the commutation relation

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$[A, A^*] = I$. Therefore, it is important to study the operator of multiplication by z and the operator of differentiation on the Fock space F^2. The commutation relation above also shows that these two operators are intimately related. The purpose of this paper is to explore this relation a little bit further.

Thus for any positive integer m we consider the space $F^{p,m}$ consisting of entire functions f such that $f^{(m)}$, the m-th order derivative of f, belongs to F^p. Because of the similarity to the way the classical Sobolev spaces are defined, we are going to call $F^{p,m}$ Fock-Sobolev spaces. See [7, 13] for other similar Sobolev spaces.

The main results of the paper can be stated as follows.

Theorem A. *Suppose $0 < p \le \infty$, m is a positive integer, and f is an entire function on $\mathbb{C}$. Then $f \in F^{p,m}$ if and only if the function $z^m f(z)$ is in F^p.*

Theorem B. *Suppose $0 < p < \infty$, m is a positive integer, and r is a positive radius. Then the following two conditions are equivalent for any positive Borel measure μ on $\mathbb{C}$.*

(a) *There exists a positive constant C such that*

$$\int_{\mathbb{C}} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p \, d\mu(z) \le C \|f\|_{p,m}^p$$

for all $f \in F^{p,m}$, where $\|f\|_{p,m}$ is the norm of the function $z^m f(z)$ in F^p.

(b) *There exists a positive constant C such that*

$$\mu(B(z,r)) \le C(1+|z|)^{mp}$$

for all $z \in \mathbb{C}$, where $B(z,r) = \{w \in \mathbb{C} : |w - z| < r\}$ is the Euclidean disk centered at z with radius r.

Theorem C. *Suppose m is a positive integer and L^p_m denote the space of Lebesgue measurable functions f on $\mathbb{C}$ such that $z^m f(z) e^{-|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. Then*

(a) *The orthogonal projection $Q_m : L^2_m \to F^{2,m}$ is a bounded projection from L^p_m onto $F^{p,m}$ when $1 \le p \le \infty$.*

(b) *If $1 \le p < \infty$ and $1/p + 1/q = 1$, then the Banach dual of $F^{p,m}$ can be identified with $F^{q,m}$ under the integral pairing*

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z) \overline{g(z)} \, e^{-|z|^2} |z|^{2m} \, dA(z).$$

(c) *If $0 < p < 1$, then the Banach dual of $F^{p,m}$ can be identified with $F^{\infty,m}$ under the integral pairing above.*

(d) *If $1 \le p_0 < p_1 \le \infty$ and $\theta \in (0,1)$, then*

$$[F^{p_0,m}, F^{p_1,m}]_\theta = F^{p,m},$$

where p is determined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Our results are based on the unweighted Fock space F^p. We could have started out with a positive parameter λ and considered the Fock spaces F^p_λ consisting of entire functions f such that the function $f(z)e^{-\lambda|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. Everything we do in the paper can be generalized to this context. No additional ideas are needed and no complications occur.

2. Preliminary Estimates

In this section we prove some preliminary estimates that will be needed in later sections. We believe that these estimates are of some independent interest for future research on Fock type spaces.

Lemma 1. *Suppose s is real and $\sigma > 0$. Then there exists a positive constant C such that*

$$\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \le Ce^x$$

for all $x \ge \sigma$. Furthermore, this holds for all $x \ge 0$ if $s \ge 0$.

Proof. Let $[s]$ denote the largest integer less than or equal to s. Then $[s] \le s < [s]+1$. For any $x > 1$ let $N = N(x)$ denote the integer such that $N < x \le N+1$. Thus for any $x > 1$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} &= \sum_{n=0}^{N-1} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} + \sum_{n=N}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \\ &\le \sum_{n=0}^{N-1} \left(\frac{x}{n+1}\right)^{[s]+1} \frac{x^n}{n!} + \sum_{n=N}^{\infty} \left(\frac{x}{n+1}\right)^{[s]} \frac{x^n}{n!}.\end{aligned}$$

If $[s] \ge 0$ and $x > 1$, it is clear that there exists a positive constant C such that

$$\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \le C \left[\sum_{n=0}^{N-1} \frac{x^{n+[s]+1}}{(n+[s]+1)!} + \sum_{n=N}^{\infty} \frac{x^{n+[s]}}{(n+[s])!}\right] \le 2C\, e^x.$$

It is easy to see that the estimate above holds for $0 \le x \le 1$ as well if $s \ge 0$, with a possibly different constant C of course.

If $[s] < 0$ and $x > 1$, we can find positive constants C_k such that

$$\begin{aligned}\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} &\le \sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^{[s]+1} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^{[s]} \frac{x^n}{n!} \\ &\le C_1 + \sum_{n=-[s]-1}^{\infty} \left(\frac{x}{n+1}\right)^{[s]+1} \frac{x^n}{n!} + \sum_{n=-[s]}^{\infty} \left(\frac{x}{n+1}\right)^{[s]} \frac{x^n}{n!} \\ &\le C_1 + C_2 \left[\sum_{n=-[s]-1}^{\infty} \frac{x^{n+[s]+1}}{(n+[s]+1)!} + \sum_{n=-[s]}^{\infty} \frac{x^{n+[s]}}{(n+[s])!}\right] \\ &= C_1 + 2C_2 e^x \le C_3 e^x.\end{aligned}$$

In the case $\sigma < 1$, the desired estimate is obvious for $\sigma \le x \le 1$ as both sides are strictly positive continuous functions. $\square$

We show that the estimate in the lemma above can be reversed.

Lemma 2. *Suppose s is real and $\sigma > 0$. Then there exists a positive constant C such that*

$$\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \ge Ce^x$$

for all $x \ge \sigma$. Furthermore, the estimate above holds for all $x \ge 0$ if $s \le 0$.

Proof. First assume that $0 \le s \le 1$. In this case, we start with $x > 1$ and let $N = N(x)$ denote the positive integer such that $N < x \le N+1$. Then

$$\left(\frac{x}{n+1}\right)^s \ge 1, \qquad 0 \le n \le N-1,$$

and

$$\left(\frac{x}{n+1}\right)^s \ge \frac{x}{n+1}, \qquad n \ge N.$$

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} &= \sum_{n=0}^{N-1} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} + \sum_{n=N}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \\
&\ge \sum_{n=0}^{N-1} \frac{x^n}{n!} + \sum_{n=N}^{\infty} \frac{x^{n+1}}{(n+1)!} \\
&= e^x - \frac{x^N}{N!} = e^x \left(1 - \frac{x^N e^{-x}}{N!}\right).
\end{aligned}$$

By Stirling's formula,

$$\lim_{x\to\infty} \left(1 - \frac{x^N e^{-x}}{N!}\right) = 1.$$

It follows easily that there exists a positive constant C such that

$$\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} \ge Ce^x$$

for all $x \ge \sigma$.

If $s > 1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^s \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^{s-[s]} \frac{x^{n+[s]}}{(n+1)^{[s]} n!} \\
&\ge \sum_{n=0}^{\infty} \left(\frac{x}{n+1+[s]}\right)^{s-[s]} \frac{x^{n+[s]}}{(n+[s])!} \\
&= \sum_{n=0}^{\infty} \left(\frac{x}{n+1}\right)^{s-[s]} \frac{x^n}{n!} - \sum_{n=0}^{[s]-1} \left(\frac{x}{n+1}\right)^{s-[s]} \frac{x^n}{n!}.
\end{aligned}$$

Since the second sum above grows polynomially and the first sum above grows at least like e^x (by what was proved in the previous paragraph), we can find a positive constant C such that

$$\sum_{n=0}^{\infty}\left(\frac{x}{n+1}\right)^s\frac{x^n}{n!}\geq Ce^x$$

for all $x\geq\sigma$.

If $s<0$, we can find positive constants C_1 and C_2 such that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{x}{n+1}\right)^s\frac{x^n}{n!} &\geq \sum_{n=-[s]}^{\infty}\left(\frac{x}{n+1}\right)^s\frac{x^n}{n!}\\ &= \sum_{n=-[s]}^{\infty}\left(\frac{x}{n+1}\right)^{s-[s]}\frac{x^{n+[s]}}{(n+1)^{[s]}n!}\\ &\geq C_1\sum_{n=-[s]}^{\infty}\left(\frac{x}{n+1+[s]}\right)^{s-[s]}\frac{x^{n+[s]}}{(n+[s])!}\\ &= C_1\sum_{n=0}^{\infty}\left(\frac{x}{n+1}\right)^{s-[s]}\frac{x^n}{n!}\\ &\geq C_2e^x\end{aligned}$$

for all $x\geq\sigma$. The last estimate above is true because $0\leq s-[s]<1$. The desired result is obvious for $x\in[0,\sigma]$ when $s\leq 0$. □

The following is a key estimate that will be used numerous times later on.

Theorem 3. *Suppose m is a nonnegative integer and $p_m(z)$ is the Taylor polynomial of e^z of order $m-1$ (with the convention that $p_0=0$). For any parameters $p>0$, $\sigma>0$, $a>0$, and $b>-(mp+2)$ we can find a positive constant C such that*

$$\int_{\mathbb{C}}|e^{z\overline{w}}-p_m(z\overline{w})|^pe^{-a|w|^2}|w|^b\,dA(w)\leq C|z|^be^{\frac{p^2}{4a}|z|^2}$$

for all $|z|\geq\sigma$. Furthermore, this holds for all z if $b\leq pm$ as well.

Proof. Let $I(z)$ denote the integral in question. Then $I(z)=I_1(z)+I_2(z)$, where

$$I_1(z)=\int_{|w|\leq 1}|e^{z\overline{w}}-p_m(z\overline{w})|^pe^{-a|w|^2}|w|^b\,dA(w),$$

and

$$I_2(z)=\int_{|w|\geq 1}|e^{z\overline{w}}-p_m(z\overline{w})|^pe^{-a|w|^2}|w|^b\,dA(w).$$

Now

$$\begin{aligned} I_1(z) &\le \int_{|w|\le 1}\left|\sum_{k=m}^{\infty}\frac{(z\overline{w})^k}{k!}\right|^p e^{-a|w|^2}|w|^b\,dA(w) \\ &= |z|^{mp}\int_{|w|\le 1}\left|\sum_{k=m}^{\infty}\frac{(z\overline{w})^{k-m}}{k!}\right|^p e^{-a|w|^2}|w|^{pm+b}\,dA(w) \\ &\le \left[\sum_{k=m}^{\infty}\frac{|z|^k}{k!}\right]^p\int_{|w|\le 1} e^{-a|w|^2}|w|^{pm+b}\,dA(w) \\ &= C[e^{|z|}-p_m(|z|)]^p. \end{aligned}$$

Note that integration in polor coordinates shows that the last integral above converges at the origin whenever $pm+b>-2$.

On the other hand,

$$I_2(z)\le 2^p(J_1(z)+J_2(z)),$$

where

$$\begin{aligned} J_1(z) &= \int_{|w|\ge 1}|p_m(z\overline{w})|^p e^{-a|w|^2}|w|^b\,dA(w) \\ &\le \int_{|w|\ge 1}\left[\sum_{k=0}^{m-1}\frac{|z|^k|w|^k}{k!}\right]^p e^{-a|w|^2}|w|^b\,dA(w) \\ &\le \left[\sum_{k=0}^{m-1}\frac{|z|^k}{k!}\right]^p\int_{|w|\ge 1} e^{-a|w|^2}|w|^{p(m-1)+b}\,dA(w) \\ &= C(p_m(|z|))^p, \end{aligned}$$

and

$$\begin{aligned} J_2(z) &= \int_{|w|\ge 1}|e^{z\overline{w}}|^p e^{-a|w|^2}|w|^b\,dA(w) \\ &= \int_{|w|\ge 1}|e^{pz\overline{w}/2}|^2 e^{-a|w|^2}|w|^b\,dA(w) \\ &= \int_{|w|\ge 1}\left|\sum_{n=0}^{\infty}\frac{(pz\overline{w}/2)^n}{n!}\right|^2 e^{-a|w|^2}|w|^b\,dA(w) \\ &= \sum_{n=0}^{\infty}\frac{(p|z|/2)^{2n}}{(n!)^2}\int_{|w|\ge 1}|w|^{2n+b}e^{-a|w|^2}\,dA(w) \\ &= \pi\sum_{n=0}^{\infty}\frac{(p|z|/2)^{2n}}{(n!)}\int_1^{\infty} r^{n+(b/2)}e^{-ar}\,dr \\ &= \frac{\pi}{a^{1+(b/2)}}\sum_{n=0}^{\infty}\frac{(p^2|z|^2/4|a|)^n}{(n!)^2}\int_a^{\infty} r^{n+(b/2)}e^{-r}\,dr. \end{aligned}$$

We can find a constant $C_1 > 0$ such that

$$\frac{1}{n!}\int_a^\infty r^{n+(b/2)}e^{-r}\,dr \le C_1(n+1)^{b/2}$$

for the possible few n that satisfies $n + (b/2) < 0$. For $n + (b/2) \ge 0$ we use Stirling's formula to find a positive constant C_2 (independent of n) such that

$$\begin{aligned}\frac{1}{n!}\int_a^\infty r^{n+(b/2)}e^{-r}\,dr &\le \frac{1}{n!}\int_0^\infty r^{n+(b/2)}e^{-r}\,dr\\ &= \frac{\Gamma(n+(b/2)+1)}{\Gamma(n+1)}\\ &\le C_2(n+1)^{b/2}.\end{aligned}$$

Therefore, there exists a positive constant C such that

$$J_2(z) \le C|z|^b\sum_{n=0}^\infty \frac{(p^2|z|^2/4a)^n}{n!}\left(\frac{n+1}{p^2|z|^2/4a}\right)^{b/2}.$$

This along with Lemma 1 shows that we find another positive constant C such that

$$J_2(z) \le C|z|^b e^{\frac{p^2}{4a}|z|^2}, \qquad |z| \ge \sigma.$$

Combining the estimates for $I_1(z)$, $J_1(z)$, and $J_2(z)$, we find another positive constant C such that

$$I(z) \le C|z|^b e^{\frac{p^2}{4a}|z|^2}, \qquad |z| \ge \sigma.$$

Finally, if $b \le pm$ as well and $|z| \le \sigma$, we have

$$\begin{aligned}I(z) &= \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|^p e^{-a|w|^2}|w|^b\,dA(w)\\ &= \int_{\mathbb{C}}\left|\sum_{n=m}^\infty \frac{[(z/\sigma)(\sigma\overline{w})]^n}{n!}\right|^p e^{-a|w|^2}|w|^b\,dA(w)\\ &\le \left|\frac{z}{\sigma}\right|^{pm}\int_{\mathbb{C}}\left|\sum_{n=m}^\infty \frac{(\sigma|w|)^n}{n!}\right|^p e^{-a|w|^2}|w|^b\,dA(w)\\ &= \left|\frac{z}{\sigma}\right|^{b}\int_{\mathbb{C}}\left[e^{\sigma|w|} - p_m(\sigma|w|)\right]^p e^{-a|w|^2}|w|^b\,dA(w)\\ &\le C|z|^b e^{\frac{p^2}{4a}|z|^2}.\end{aligned}$$

This completes the proof of the theorem. □

3. The Fock-Sobolev spaces $F^{p,m}$

The purpose of this section is to show that, for an entire function f and a positive integer m, the function $f^{(m)}(z)$ belongs to F^p if and only if the function $z^m f(z)$ is in F^p.

For any $0 < p \le \infty$ let L^p_* denote the space of measurable functions f such that the function $f(z)e^{-|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. If $0 < p < \infty$, the norm in L^p_* is defined by

$$\|f\|_p^p = \frac{p}{2\pi}\int_{\mathbb{C}} \left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p \, dA(z).$$

For $p = \infty$, the norm in L^∞_* is defined by

$$\|f\|_\infty = \sup_{z\in\mathbb{C}} |f(z)|e^{-\frac{1}{2}|z|^2}.$$

Recall that F^2 is a closed subspace of L^2_*, and the orthogonal projection $P : L^2_* \to F^2$ is given by

$$Pf(z) = \frac{1}{\pi}\int_{\mathbb{C}} e^{z\overline{w}} f(w)e^{-|w|^2} \, dA(w).$$

It is well known that for $1 \le p \le \infty$ the Fock space F^p is a closed subspace of L^p_* and the Fock projection P above is a bounded projection from L^p_* onto F^p. See [9] for example.

Lemma 4. *Suppose $\lambda > 0$ and $0 < p \le 1$. There exists a positive constant C such that*

$$\int_{\mathbb{C}} |f(z)|e^{-\frac{\lambda}{2}|z|^2} \, dA(z) \le C\left[\int_{\mathbb{C}} \left|f(z)e^{-\frac{\lambda}{2}|z|^2}\right|^p \, dA(z)\right]^{1/p}$$

for all entire functions f.

Proof. It is well known that $|f(z)| \le \|f\|_p e^{|z|^2/2}$ for all entire functions f and all $z \in \mathbb{C}$. See [9] or [14]. By a simple of change of variables, this can easily be generalized to weighted Fock spaces with the Gaussian weight $e^{-\lambda|z|^2}$. More specifically, we have

$$|f(z)|e^{-\lambda|z|^2/2} \le \left[\frac{p\lambda}{2\pi}\int_{\mathbb{C}} \left|f(z)e^{-\frac{\lambda}{2}|z|^2}\right|^p \, dA(z)\right]^{\frac{1}{p}} \tag{1}$$

for all entire functions f and all $z \in \mathbb{C}$. See [14] and [15] for example.

The desired inequality now follows easily by writing

$$|f(z)|e^{-\frac{\lambda}{2}|z|^2} = \left|f(z)e^{-\lambda|z|^2/2}\right|^p \left|f(z)e^{-\lambda|z|^2/2}\right|^{1-p}$$

and estimating the factor $|f(z)e^{-\lambda|z|^2/2}|^{1-p}$ using (1). □

As a consequence of the lemma above, we have $F^p \subset F^1$ when $0 < p \le 1$. More generally, we always have $F^p \subset F^q$ whenever $0 < p \le q \le \infty$. See [9] and [14] for example.

Lemma 5. *Suppose $0 < p \le \infty$, m is a positive integer, and f is an entire function. If the function $z^m f(z)$ is in F^p, then so is $f^{(m)}$.*

Proof. If the function $w^m f(w)$ is in F^p, then the following reproducing formula holds:

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}} e^{z\overline{w}} f(w) e^{-|w|^2}\, dA(w), \qquad z \in \mathbb{C}.$$

The convergence of the integral above follows from (1). Differentiating under the integral sign m times, we obtain

$$f^{(m)}(z) = \frac{1}{\pi}\int_{\mathbb{C}} e^{z\overline{w}}\, \overline{w}^m f(w) e^{-|w|^2}\, dA(w). \tag{2}$$

In other words, we have $f^{(m)} = Pg$, where $g(z) = \overline{z}^m f(z)$. Since $g \in L^p_*$ and P is a projection of L^p_* onto F^p for $1 \le p \le \infty$, the desired result is clear when $1 \le p \le \infty$.

When $0 < p < 1$, we note from (2) that

$$|f^{(m)}(z)| \le \frac{1}{\pi}\int_{\mathbb{C}} |w^m f(w) e^{\overline{z}w}| e^{-|w|^2}\, dA(w).$$

By Lemma 4, there exists a positive constant C, independent of f, such that

$$|f^{(m)}(z)|^p \le C\int_{\mathbb{C}} \left| w^m f(w) e^{\overline{z}w} e^{-|w|^2}\right|^p\, dA(w).$$

It follows from this and Fubini's theorem that we can estimate the integral

$$I = \int_{\mathbb{C}} |f^{(m)}(z) e^{-|z|^2/2}|^p\, dA(z)$$

as follows.

$$\begin{aligned} I &\le C\int_{\mathbb{C}} e^{-p|z|^2/2}\, dA(z) \int_{\mathbb{C}} |w^m f(w)|^p e^{-p|w|^2} |e^{\overline{z}w}|^p\, dA(w) \\ &= C\int_{\mathbb{C}} |w^m f(w)|^p e^{-p|w|^2}\, dA(w) \int_{\mathbb{C}} |e^{\frac{p}{2}w\overline{z}}|^2 e^{-\frac{p}{2}|z|^2}\, dA(z) \\ &= \frac{2\pi C}{p}\int_{\mathbb{C}} |w^m f(w)|^p e^{-p|w|^2} e^{\frac{p}{2}|w|^2}\, dA(w) \\ &= \frac{2\pi C}{p}\int_{\mathbb{C}} |w^m f(w) e^{-|w|^2/2}|^p\, dA(w). \end{aligned}$$

This shows that $f^{(m)} \in F^p$ and completes the proof of the lemma. □

Lemma 6. *Suppose $0 < p \le \infty$, m is a positive integer, and f is an entire function. If the function $f^{(m)}(z)$ is in F^p, then so is the function $z^m f(z)$.*

Proof. Since $f^{(m)}$ is in F^p, we have the reproducing formula

$$f^{(m)}(z) = \frac{1}{\pi}\int_{\mathbb{C}} e^{z\overline{w}} f^{(m)}(w) e^{-|w|^2}\, dA(w), \qquad z \in \mathbb{C}.$$

Again, the convergence of the integral above follows from (1). Integrating from 0 to z repeatedly for m times, we obtain

$$f(z) - f_m(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{z\overline{w}} - p_m(z\overline{w})}{\overline{w}^m}\, f^{(m)}(w) e^{-|w|^2}\, dA(w), \tag{3}$$

where

$$f_m(z) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} z^k, \quad p_m(z) = \sum_{k=0}^{m-1} \frac{z^k}{k!}.$$

We first consider the case $p = \infty$. In this case, it follows from (3) and Theorem 3 that there is a positive constant C such that

$$\begin{aligned} |f(z) - f_m(z)| &\leq \frac{1}{\pi} \|f^{(m)}\|_\infty \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|\, e^{-|w|^2/2} |w|^{-m}\, dA(w) \\ &\leq C \|f^{(m)}\|_\infty |z|^{-m} e^{|z|^2/2} \end{aligned}$$

for all z. Since every polynomial belongs to F^∞, we deduce from this that the function $z^m f(z)$ is in F^∞ whenever $f^{(m)}(z)$ is in F^∞.

Next we consider the case $p = 1$. It follows from (3) and Fubini's theorem that

$$\begin{aligned} &\int_{\mathbb{C}} |z|^m |f(z) - f_m(z)| e^{-|z|^2/2}\, dA(z) \\ \leq\ & \frac{1}{\pi} \int_{\mathbb{C}} |z|^m e^{-|z|^2/2}\, dA(z) \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})| |f^{(m)}(w)| e^{-|w|^2} |w|^{-m}\, dA(w) \\ =\ & \frac{1}{\pi} \int_{\mathbb{C}} e^{-|w|^2} |f^{(m)}(w)| |w|^{-m}\, dA(w) \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})| e^{-|z|^2/2} |z|^m\, dA(z). \end{aligned}$$

This together with Theorem 3 shows that there is a positive constant C such that

$$\int_{\mathbb{C}} |z|^m |f(z) - f_m(z)| e^{-|z|^2/2}\, dA(z) \leq C \int_{\mathbb{C}} |f^{(m)}(w)| e^{-|w|^2/2}\, dA(w).$$

Thus $z^m f(z)$ is in F^1 whenever $f^{(m)}(z)$ is in F^1.

With the help of complex interpolation we conclude that the function $z^m f(z)$ is in F^p whenever the function $f^{(m)}(z)$ is in F^p, where $1 \leq p \leq \infty$.

Finally, if $0 < p < 1$, we note from (3) that

$$|f(z) - f_m(z)| \leq \frac{1}{\pi} \int_{\mathbb{C}} \left| \frac{e^{\overline{z}w} - p_m(\overline{z}w)}{w^m} f^{(m)}(w) e^{-|w|^2} \right| dA(w).$$

Again, by Lemma 4 (with $\lambda = 2$), there exists a positive constant C such that

$$|f(z) - f_m(z)|^p \leq C \int_{\mathbb{C}} \left| \frac{e^{\overline{z}w} - p_m(\overline{z}w)}{w^m} f^{(m)}(w) e^{-|w|^2} \right|^p dA(w).$$

This together with Fubini's theorem shows that the integral

$$J = \int_{\mathbb{C}} |z^m (f(z) - f_m(z)) e^{-|z|^2/2}|^p\, dA(z)$$

satisfies the following estimates:

$$\begin{aligned} J &\leq C \int_{\mathbb{C}} |z|^{mp} e^{-\frac{p}{2}|z|^2}\, dA(z) \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|^p |f^{(m)}(w)|^p e^{-p|w|^2} \frac{dA(w)}{|w|^{mp}} \\ &= C \int_{\mathbb{C}} |f^{(m)}(w)|^p e^{-p|w|^2} \frac{dA(w)}{|w|^{mp}} \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|^p e^{-\frac{p}{2}|z|^2} |z|^{mp}\, dA(z). \end{aligned}$$

By Theorem 3 there exists another positive constant C such that

$$\int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|^p e^{-\frac{p}{2}|z|^2} |z|^{mp}\, dA(z) \le C|w|^{mp} e^{\frac{p}{2}|w|^2}$$

for all $w \in \mathbb{C}$. This shows that

$$J \le C \int_{\mathbb{C}} |f^{(m)}(w) e^{-\frac{1}{2}|w|^2}|^p\, dA(w)$$

for another positive constant C that is independent of f. Therefore, the function $z^m(f(z) - f_m(z))$ is in F^p. Since every polynomial is in F^p, we conclude that the function $z^m f(z)$ is in F^p. This completes the proof of the lemma. □

Combining Lemmas 5 and 6, we have now proved the following theorem, the main result of the section.

Theorem 7. *Suppose $0 < p \le \infty$, m is a positive integer, and f is an entire function. Then $f^{(m)}$ is in F^p if and only if the function $z^m f(z)$ is in F^p.*

4. Carleson Measures

Since $F^{p,m}$ consists of entire functions f such that $z^m f(z)$ is in F^p, it is natural to consider the following norm on $F^{p,m}$ when $0 < p < \infty$:

$$\|f\|_{p,m}^p = c(p,m) \int_{\mathbb{C}} |z^m f(z) e^{-|z|^2/2}|^p\, dA(w), \tag{4}$$

where

$$c(p,m) = \frac{(p/2)^{(mp/2)+1}}{\pi\, \Gamma((mp/2)+1)}$$

is a normalizing constant so that the constant function 1 has norm 1 in $F^{p,m}$. By the open mapping theorem, we have

$$\|f\|_{p,m} \sim |f(0)| + \cdots + |f^{(m-1)}(0)| + \|f^{(m)}\|_{F^p}.$$

It is more convenient for us to use the norm $\|f\|_{p,m}$ defined in (4).

Proposition 8. *The Hilbert space $F^{2,m}$ possesses the following orthonormal basis:*

$$e_n(z) = \sqrt{\frac{\Gamma(m+1)}{\Gamma(n+m+1)}}\, z^n = \sqrt{\frac{m!}{(n+m)!}}\, z^n, \qquad n = 0, 1, 2, \cdots.$$

Consequently, the reproducing kernel of $F^{2,m}$ is given by

$$K_m(z,w) = \sum_{n=0}^{\infty} \frac{m!}{(n+m)!} (z\overline{w})^n = \frac{m!}{(z\overline{w})^m} \left[e^{z\overline{w}} - p_m(z\overline{w}) \right],$$

where p_m is the Taylor polynomial of e^z of order $m-1$.

Proof. The first assertion follows from elementary calculations using polar coordinates. The second assertion follows from the fact that

$$K_m(z,w) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)},$$

where $\{e_n\}$ is any orthonormal basis of $F^{2,m}$. □

Lemma 9. *Suppose $p > 0$, $t > 0$, and $\lambda > 0$. There exists a positive constant C such that*

$$\left|f(z)e^{-\frac{\lambda}{2}|z|^2}\right|^p \le C\int_{|w-z|<t}\left|f(w)e^{-\frac{\lambda}{2}|w|^2}\right|^p dA(w)$$

for all entire functions f and all $z \in \mathbb{C}$.

Proof. This is well known. See [8] or [15]. □

We now prove the main result of this section.

Theorem 10. *Suppose $0 < p < \infty$, m is a positive integer, r is a positive radius, and μ is a positive Borel measure on $\mathbb{C}$. Then the following two conditions are equivalent.*

(a) *There exists a positive constant C such that*

$$\int_{\mathbb{C}}\left|f(z)e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) \le C\|f\|_{p,m}^p \tag{5}$$

for all $f \in F^{p,m}$.

(b) *There exists a positive constant C such that*

$$\mu(B(a,r)) \le C(1+|a|)^{mp} \tag{6}$$

for all $a \in \mathbb{C}$, where $B(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ is the Euclidean disk centered at a with radius r.

Proof. First assume that there is a positive constant C such that (5) holds for all $f \in F^{p,m}$. Taking $f = 1$ shows that $\mu(S) < \infty$ for any compact set S.

Fix any complex number a and let

$$f(z) = [e^{z\overline{a}} - p_m(z\overline{a})]/z^m$$

in (5). Then it follows from Theorem 3 that there exists another constant $C > 0$, independent of a, such that

$$\int_{\mathbb{C}}\left|\frac{e^{z\overline{a}} - p_m(z\overline{a})}{z^m}\, e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) \le Ce^{\frac{p}{2}|a|^2}.$$

In particular,

$$\int_{|z-a|<r}\left|\frac{e^{z\overline{a}} - p_m(z\overline{a})}{z^m}\, e^{-\frac{1}{2}|z|^2}\right|^p d\mu(z) \le Ce^{\frac{p}{2}|a|^2}.$$

If $|a| > 2r$, then $|z|^m$ is comparable to $(1+|a|)^m$ for $|z-a| < r$. So there is another positive constant C such that

$$\int_{|z-a|<r}|e^{z\overline{a}}|^p|1 - e^{-z\overline{a}}p_m(z\overline{a})|^p e^{-\frac{p}{2}|z|^2}\, d\mu(z) \le C(1+|a|)^{mp}e^{\frac{p}{2}|a|^2}$$

for all $|a| > 2r$. It is easy to show that

$$\lim_{a\to\infty, |z-a|<r}(1 - e^{-z\overline{a}}p_m(z\overline{a})) = 1.$$

Thus we can find another constant $C > 0$ with

$$\int_{|z-a|<r} |e^{z\overline{a}}|^p e^{-\frac{p}{2}|z|^2}\, d\mu(z) \le C(1+|a|)^{mp} e^{\frac{p}{2}|a|^2}$$

for large $|a|$. But this is clearly true (with a different constant C) for smaller $|a|$ as well. So we can find another constant $C > 0$ such that

$$\int_{|z-a|<r} e^{-\frac{p}{2}|a|^2} |e^{z\overline{a}}|^p e^{-\frac{p}{2}|z|^2}\, d\mu(z) \le C(1+|a|)^{mp}$$

for all $a \in \mathbb{C}$. Completing a square in the exponent, we can rewrite the inequality above as

$$\int_{|z-a|<r} e^{-\frac{p}{2}|z-a|^2}\, d\mu(z) \le C(1+|a|)^{mp},$$

from which we deduce that

$$\mu(B(a,r)) \le Ce^{-\frac{p}{2}r^2}(1+|a|)^{mp}$$

for all $a \in \mathbb{C}$. This shows that condition (a) implies condition (b).

Next we assume that there exists a constant $C > 0$ such that (6) holds for all $a \in \mathbb{C}$. We proceed to estimate the integral

$$I(f) = \int_{\mathbb{C}} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p d\mu(z)$$

for any function $f \in F^{p,m}$.

For any positive s let Q_s denote the following square in $\mathbb{C}$ with vertices 0, s, si, and $s+si$:

$$Q_s = \{z = x+iy : 0 < x \le s, 0 < y \le s\}.$$

Let Z denote the set of all integers and consider the lattice

$$Z_s^2 = \{ns + ims : n \in Z, m \in Z\}.$$

It is clear that

$$\mathbb{C} = \bigcup \{Q_s + a : a \in Z_s^2\}$$

is decomposition of $\mathbb{C}$ into disjoint squares of side length s. Thus

$$I(f) = \sum_{a \in Z_s^2} \int_{Q_s+a} \left| f(z) e^{-\frac{1}{2}|z|^2} \right|^p d\mu(z).$$

We fix positive numbers s and t such that $t + \sqrt{s} = r$. By Lemma 9 there exists a constant C such that

$$|f(z)|^p e^{-\frac{p}{2}|z|^2} \le C \int_{|w-z|<t} |f(w) e^{-\frac{1}{2}|w|^2}|^p\, dA(w)$$

for all $z \in \mathbb{C}$. From this we easily deduce that

$$|f(z) e^{-\frac{1}{2}|z|^2}|^p \le \frac{C}{(1+|z|)^{mp}} \int_{|w-z|<t} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p\, dA(w)$$

for all $z \in \mathbb{C}$, where C is another positive constant. Now if $z \in Q_s + a$, where $a \in Z_s^2$, then $B(z,t) \subset B(a,r)$ by the triangle inequality, and $1+|z|$ is comparable to $1+|a|$. It follows that

$$|f(z)e^{-\frac{1}{2}|z|^2}|^p \le \frac{C}{(1+|a|)^{mp}} \int_{|w-a|<r} |w^m f(w) e^{-\frac{1}{2}|w|^2}|^p \, dA(w),$$

where C is another positive constant. Therefore,

$$I(f) \le C \sum_{a \in Z_s^2} \frac{\mu(B(a,r))}{(1+|a|)^{mp}} \int_{B(a,r)} \left| w^m f(w) e^{-\frac{1}{2}|w|^2} \right|^p \, dA(w).$$

Combining this with the assumption in (6), we find another positive constant C such that

$$I(f) \le C \sum_{a \in Z_s^2} \int_{B(a,r)} \left| w^m f(w) e^{-\frac{1}{2}|w|^2} \right|^p \, dA(w).$$

It is clear that there exists a positive integer N such that each point in the complex plane belongs to at most N of the disks $B(a,r)$, where $a \in Z_s^2$. It follows that

$$I(f) \le CN \int_{\mathbb{C}} \left| w^m f(w) e^{-\frac{1}{2}|w|^2} \right|^p \, dA(w).$$

Since C and N are independent of f, this shows that condition (b) implies condition (a). □

The following is the "little oh" version of Theorem 10. The proof is similar to that of Theorem 10 and we omit the details here.

Theorem 11. *Suppose $0 < p < \infty$, m is a positive integer, r is a positive radius, and μ is a positive Borel measure on $\mathbb{C}$. Then*

$$\lim_{a \to \infty} \frac{\mu(B(a,r))}{(1+|a|)^{mp}} = 0$$

if and only if $F^{p,m} \subset L^p(\mathbb{C}, d\mu)$ and the inclusion is compact.

Note that compact inclusion in the theorem above means the following: whenever $\{f_n\}$ is a bounded sequence in $F^{p,m}$ that converges to 0 uniformly on compact sets we have

$$\lim_{n \to \infty} \int_{\mathbb{C}} \left| f_n(z) e^{-\frac{1}{2}|z|^2} \right|^p \, d\mu(z) = 0.$$

When $1 < p < \infty$, this is consistent with the definition of compact linear operators in the theory of Banach spaces.

5. Duality and complex interpolation

Recall that the Fock-Sobolev space $F^{p,m}$ consists of entire functions f such that the function $z^m f(z) e^{-|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. More generally, we let L_m^p denote the space of Lebesgue measurable functions f on the complex plane such that the

function $z^m f(z)e^{-|z|^2/2}$ is in $L^p(\mathbb{C}, dA)$. When $0 < p < \infty$, we use the following norm in L^p_m:

$$\|f\|^p_{p,m} = c(m,p) \int_{\mathbb{C}} \left| z^m f(z) e^{-|z|^2/2} \right|^p \, dA(z),$$

where $c(m,p)$ is a normalizing constant defined earlier. When $p = \infty$, we use the following norm in L^∞_m:

$$\|f\|_{\infty,m} = \sup\{|z^m f(z) e^{-|z|^2/2}| : z \in \mathbb{C}\}.$$

The space L^2_m is a Hilbert space and the inner product on it is given by

$$\langle f, g\rangle_m = \frac{1}{m!\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-|z|^2} |z|^{2m} \, dA(z). \tag{7}$$

The Fock-Sobolev space $F^{2,m}$ is a closed subspace of L^2_m, and the orthogonal projection $Q_m : L^2_m \to F^{2,m}$ is given by

$$Q_m f(z) = \frac{1}{m!\pi} \int_{\mathbb{C}} f(w) K_m(z,w) e^{-|w|^2} |w|^{2m} \, dA(w), \tag{8}$$

where $K_m(z,w)$ is the reproducing kernel of $F^{2,m}$ given in Proposition 8.

Theorem 12. *For each $1 \le p \le \infty$ the integral operator Q_m in (8) is a bounded projection from L^p_m onto $F^{p,m}$.*

Proof. It suffices to show that the linear operator Q_m is bounded on L^p_m for $1 \le p \le \infty$. The reproducing property of K_m then shows that Q_m is a projection from L^p_m onto $F^{p,m}$.

We first consider the case when $p = \infty$. If $f \in L^\infty_m$, then

$$\begin{aligned} |Q_m f(z)| &\le \frac{1}{m!\pi} \int_{\mathbb{C}} |f(w)| \frac{|e^{z\overline{w}} - p_m(z\overline{w})|}{|zw|^m} e^{-|w|^2} |w|^{2m} \, dA(w) \\ &\le \frac{1}{m!\pi |z|^m} \|f\|_{\infty,m} \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})| e^{-|w|^2/2} \, dA(w). \end{aligned}$$

By Theorem 3, there exists a constant $C > 0$, independent of z and f, such that

$$|z^m Q_m f(z)| \le C\|f\|_{\infty,m} e^{|z|^2/2}, \qquad z \in \mathbb{C}.$$

This shows that $\|Q_m f\|_{\infty,m} \le C\|f\|_{\infty,m}$ for all $f \in L^\infty_m$. Thus Q_m is bounded on L^∞_m.

We next consider the case when $p = 1$. So let $f \in L^1_m$ and consider the integral

$$I = m!\pi \int_{\mathbb{C}} |z^m Q_m f(z) e^{-|z|^2/2}| \, dA(z).$$

By Fubini's theorem,

$$\begin{aligned} I &\le \int_{\mathbb{C}} |z|^m e^{-|z|^2/2} \, dA(z) \int_{\mathbb{C}} \frac{|f(w)||e^{z\overline{w}} - p_m(z\overline{w})|}{|zw|^m} e^{-|w|^2} |w|^{2m} \, dA(w) \\ &= \int_{\mathbb{C}} |w^m f(w)| e^{-|w|^2} \, dA(w) \int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})| e^{-|z|^2/2} \, dA(z). \end{aligned}$$

According to Theorem 3 again, there exists a positive constant C, independent of f, such that

$$I \le C \int_{\mathbb{C}} |w^m f(w)| e^{-|w|^2/2}\, dA(w).$$

This shows that Q_m is bounded on L^1_m.

It is well known that the scale of spaces L^p_m interpolate in the usual way:

$$[L^{p_0}_m, L^{p_1}_m]_\theta = L^p_m,$$

where $1 \le p_0 < p_1 \le \infty$, $\theta \in (0,1)$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

See [12] for example. Therefore, the boundedness of the linear operator Q_m on L^∞_m and L^1_m imply the boundedness of Q_m on L^p_m for all $1 \le p \le \infty$. □

Theorem 13. *Suppose $1 \le p < \infty$ and $1/p + 1/q = 1$. Then $(F^{p,m})^*$, the Banach dual of $F^{p,m}$, can be identified with $F^{q,m}$ under the integral pairing in (7).*

Proof. That every function in $F^{q,m}$ induces a bounded linear functional on $F^{p,m}$ via the integral pairing in (7) follows from the Cauchy-Schwarz inequality.

On the other hand, if F is a bounded linear functional on $F^{p,m}$, then according to the Hahn-Banach extension theorem, F can be extended (without increasing its norm) to a bounded linear functional on L^p_m. By the usual duality of L^p_m spaces, there exists some $h \in L^q_m$ such that

$$F(f) = \langle f, h\rangle_m, \qquad f \in F^{p,m}.$$

By Theorem 12, Q_m is a bounded projection from L^p_m onto $F^{p,m}$. So if we let $g = Q_m(h)$, then $g \in F^{p,m}$ and

$$F(f) = \langle f, h\rangle_m = \langle Q_m(f), h\rangle_m = \langle f, Q_m(h)\rangle_m = \langle f, g\rangle_m$$

for all $f \in F^{p,m}$. This completes the proof of the theorem. □

A similar result holds for $0 < p < 1$.

Theorem 14. *Suppose $0 < p < 1$ and m is a positive integer. Then the dual space of $F^{p,m}$ can be identified with $F^{\infty,m}$ under the integral pairing in (7).*

Proof. First suppose that $g \in F^{\infty,m}$ and

$$F(f) = \int_{\mathbb{C}} f(z)\overline{g(z)}\, e^{-|z|^2} |z|^{2m}\, dA(z), \qquad f \in F^{p,m}.$$

Then

$$|F(f)| \le \|g\|_{\infty,m} \int_{\mathbb{C}} |z^m f(z) e^{-|z|^2/2}|\, dA(z),$$

and an application of Lemma 4 to the function $z^m f(z)$ yields

$$|F(f)| \le C\|g\|_{\infty,m}\|f\|_{p,m}, \qquad f \in F^{p,m},$$

where C is a positive constant independent of f. This shows that F defines a bounded linear functional on $F^{p,m}$.

Conversely, if F is a bounded linear functional on $F^{p,m}$, then for any $f \in F^{p,m}$ we deduce from the reproducing formula

$$f(z) = \frac{1}{m!\pi} \int_{\mathbb{C}} f(w) K_m(z,w) e^{-|w|^2} |w|^{2m} \, dA(w)$$

that

$$F(f) = \frac{1}{m!\pi} \int_{\mathbb{C}} f(w) \overline{g(w)} \, e^{-|w|^2} |w|^{2m} \, dA(w).$$

where

$$g(w) = \overline{F(K_m(\,\cdot\,,w))}.$$

It is clear that g is entire and

$$\begin{aligned} |g(w)| &\leq \|F\| \|K_m(\,\cdot\,,w)\|_{p,m} \\ &= \|F\| \left[\int_{\mathbb{C}} \left| \frac{e^{z\overline{w}} - p_m(z\overline{w})}{(z\overline{w})^m} \, z^m e^{-\frac{1}{2}|z|^2} \right|^p \, dA(z) \right]^{\frac{1}{p}} \\ &= \frac{\|F\|}{|w|^m} \left[\int_{\mathbb{C}} |e^{z\overline{w}} - p_m(z\overline{w})|^p e^{-\frac{p}{2}|z|^2} \, dA(z) \right]^{\frac{1}{p}}. \end{aligned}$$

According to Theorem 3, there is a positive constant C such that

$$|w^m g(w)| \leq C \left[e^{\frac{p}{2}|w|^2} \right]^{\frac{1}{p}}, \qquad w \in \mathbb{C}.$$

This shows that $g \in F^{\infty,m}$ and completes the proof of the theorem. □

Theorem 15. *Suppose* $1 \leq p_0 < p_1 \leq \infty$, $\theta \in (0,1)$, *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then $[F^{p_0,m}, F^{p_1,m}]_\theta$, *the complex interpolation space between* $F^{p_0,m}$ *and* $F^{p_1,m}$, *can be identified with* $F^{p,m}$.

Proof. That $[F^{p_0,m}, F^{p_1,m}]_\theta \subset F^{p,m}$ follows from the definition of complex interpolation, the fact that each $F^{p_k,m}$ is a closed subspace of $L^{p_k}_m$, and the fact that $[L^{p_0}_m, L^{p_1}_m]_\theta = L^p_m$.

Conversely, if $f \in F^{p,m} \subset L^p_m$, then it follows from the fact that $[L^{p_0}_m, L^{p_1}_m]_\theta = L^p_m$ there exists a function $F(z,\zeta)$, where $z \in \mathbb{C}$ and $0 \leq \mathrm{Re}\,(\zeta) \leq 1$, and a positive constant C, such that

(a) $F(z,\theta) = f(z)$ for all $z \in \mathbb{C}$.
(b) $\|F(\,\cdot\,,\zeta)\|_{p_0,m} \leq C$ for all $\mathrm{Re}\,(\zeta) = 0$.
(c) $\|F(\,\cdot\,,\zeta)\|_{p_1,m} \leq C$ for all $\mathrm{Re}\,(\zeta) = 1$.

Let $G(z,\zeta) = Q_m F(z,\zeta)$, where the projection Q_m is applied with respect to the variable z. Then for any fixed ζ the function $G(z,\zeta)$ is entire in z, and the boundedness of the projection Q_m on $L^{p_k}_m$ shows that there is another positive constant C such that

(a) $G(z,\theta) = f(z)$ for all $z \in \mathbb{C}$.
(b) $\|G(\,\cdot\,,\zeta)\|_{p_0,m} \leq C$ for all $\mathrm{Re}\,(\zeta) = 0$.
(c) $\|G(\,\cdot\,,\zeta)\|_{p_0,m} \leq C$ for all $\mathrm{Re}\,(\zeta) = 1$.

This shows that $f \in [F^{p_0,m}, F^{p_1,m}]_\theta$, and completes the proof of the theorem. $\square$

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