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State University of New York

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Technical Notes

John Bailey Jones

1 Optimization Under Uncertainty

While dynamic optimization can be challenging even under perfect foresight, introducing uncertainty generates a new set of problems. In particular, people’s information sets—knowledge—constantly expand over time. For example, in the year 1979 a worker might not yet know what he will earn in 1980; in the year 1981, he will have found out. Since we like to characterize dynamic problems as starting at some point in time (typically time 0), this would seem to pose a problem. We should expect people to make decisions with all the information they have available. But how can we characterize a sequence of actions on a time-0 basis when at time 0 we don’t know all the information that will be used in future decisions? To be concrete, our worker’s decision on how much to save today depends in part on how much he will be saving in the future. But the worker’s choice of future savings will depend on information, such as realized future income, that is not known today.

Fortunately, it turns out that under conditions similar to those that allow us to attack perfect foresight problems in a sequential manner, we can attack stochastic problems in a sequential manner. The key insight is to turn the problem from a choice of values into a choice of **functions**. For example, our worker in 1979 chooses the level of 1979 saving, but a **contingency plan** for setting saving in 1980. This contingency plan might include the following instructions: “if income in 1980 is \$30,000, I will save \$10,000, while if income is \$35,000, I will save \$12,500.” Since the contingency plan will be a function of the stochastic elements that affect our agent, if the agent can predict the stochastic elements of his environment, he can predict the actions he will take under his contingency plan. And that gives him something to base his current decision on.

Let’s study this problem more formally. We will first consider a general structure where the sequential approach makes a lot of sense. We will then digress to see what happens when the dynamic problem does not satisfy this structure. Finally, we will put some restrictions on the stochastic environment that allow us to use dynamic programming, i.e., allow us to break a multi-period problem into a series of one-period problems.

Readers seeking more background should consult Sargent (1987, chapter 1). Rigorous discussions of dynamic programming include Stokey and Lucas (1989) and Bertsekas (1987).

1.1 A Time-consistent Problem

Let's begin by considering the following three-period problem:

$$\begin{aligned}
 \max \quad & E \left(R(x_1, k_1, \eta_1) + \beta U(x_2, k_2, \eta_2) + \beta^2 W(k_3, \eta_3) \mid I_1 \right) & \text{(PR1)} \\
 \text{s.t.} \quad & k_2 = G(x_1, k_1, \eta_2), \\
 & k_3 = G(x_2, k_2, \eta_3), \\
 & k_1 \text{ given,}
 \end{aligned}$$

where: x_j is a control variable, such as consumption; k_j is an endogenous state variable, such as a capital stock; η_j is a random exogenous state variable, such as endowments in an endowment economy; I_j is the information set agents use for making predictions; and $E(\cdot \mid I_j)$ yields expectations conditional on set I_j . The discount factor β is restricted to lie in the interval $(0, 1)$. Note that the transition function relates the future endogenous state variable to decisions made in the previous period, prior state variables, and realizations of the exogenous state variable. While the problem is pretty abstract, we could make it even more general. For example, we could make the return in period 3, $W(\cdot)$ also depend on some sort of control variable, and we could allow the transition function $G(\cdot)$ to vary over time. One way in which the problem is restrictive, however, is that we force it to be recursive: returns in any given period cannot depend on control variables chosen in future periods. This will turn out to be critical.

To solve this problem, we choose over contingency plans by picking functions

$$x_j = h_j(I_j), \quad j = 1, 2.$$

Then the problem given by (PR1) becomes

$$\begin{aligned}
 \max_{\{h_1(\cdot), h_2(\cdot)\}} \quad & E \left(R(h_1(I_1), k_1, \eta_1) + \beta U(h_2(I_2), k_2, \eta_2) + \beta^2 W(k_3, \eta_3) \mid I_1 \right) \\
 \text{s.t.} \quad & k_2 = G(h_1(I_1), k_1, \eta_2), \\
 & k_3 = G(h_2(I_2), k_2, \eta_3), \\
 & k_1 \text{ given.}
 \end{aligned}$$

While this means that x_2 is not known at time 1, since x_2 is a function of I_2 , if the agent can predict I_2 , she can predict x_2 . Let's now turn to finding the function $h_2(\cdot)$. Since x_2

does not enter into $R(\cdot)$, we pick $h_2(\cdot)$ to satisfy

$$\max_{\{h_2(\cdot)\}} \int \left[\beta U(h_2(I_2), k_2, \eta_2) + \beta^2 E(W[G(h_2(I_2), k_2, \eta_3), \eta_3] | I_2) \right] dF(I_2 | I_1).$$

In addition to explicitly writing the time-1 expectation as an integral, we have used the law of iterated expectations to express the time-1 expectation of $W(\cdot)$ as the time-1 expectation of the time-2 expectation of $W(\cdot)$. Note that what we are doing is picking $h_2(\cdot)$ to maximize the integral of our objective function (the stuff in the square brackets) over different realizations of I_2 . (As a matter of notation, integrals with an unspecified interval are taken over the entire domain of the variable of integration, in this case, I_2 .)

Before we start looking over some space of functions, let's suppose that we find the value of x_2 that is optimal for a particular value of I_2 . This allows us to define a function—there is no loss in generality in assuming that it is single-valued—that we will call $h_2^*(I_2)$:

$$h_2^*(I_2) = \underset{x_2}{\operatorname{argmax}} U(x_2, k_2, \eta_2) + \beta E(W(G(x_2, k_2, \eta_3), \eta_3) | I_2),$$

for each value of I_2 . But if we have maximized our objective function for each value of I_2 , then surely we have maximized the weighted average (the integral) of our objective function over all values of I_2 . The only reason $h_2^*(\cdot)$ would not be optimal is if there were constraints on $h_2(\cdot)$ between different values of I_2 . Since there are no such constraints, $h_2^*(\cdot)$ is the maximizing function. Conditional (on I_2) maximization produces unconditional maximization. What this means is that our initial optimization problem (PR1), which involves a time 1 expectation, is maximized by a time 2 contingency plan, $h_2^*(I_2)$, that fully utilizes all the information available at time 2.

1.2 Time Inconsistency

The results of the previous section hold under a variety of generalizations. We could allow $U(\cdot)$ to depend on k_3 , so that current returns depend on future states, or we could allow $W(\cdot)$ to depend on x_2 , so that future returns depend on current controls. What we cannot do is allow current returns to depend—directly or through k —on future controls. To see this point, consider

$$\begin{aligned}
& \max_{\{h_1(\cdot), h_2(\cdot)\}} E \{ R(h_1(I_1), k_1, \eta_1, E(h_2(I_2)|I_1)) + \beta U(h_2(I_2), k_2, \eta_2) \\
& \quad + \beta^2 W(k_3, \eta_3) | I_1 \} \\
& \text{s.t. } k_2 = G(h_1(I_1), k_1, \eta_2), \\
& \quad k_3 = G(h_2(I_2), k_2, \eta_3), \\
& \quad k_1 \text{ given.}
\end{aligned} \tag{PR2}$$

The current problem, (PR2), differs from the previous one, (PR1) in that we allow $R(\cdot)$ to depend on the expected value of x_2 . It immediately follows that $h_2^*(\cdot)$ is not a solution to (PR2), because $h_2^*(\cdot)$ ignores the effect of $E(h_2(I_2)|I_1)$ on $R(\cdot)$. Conditional maximization does not produce unconditional maximization.

This sort of time-inconsistency appears in Kydland and Prescott's (1977) famous paper on optimal government policy. Kydland and Prescott consider the case where a government tries to maximize the discounted sum of its citizens' utility. They point out that the private sector's current actions, and thus its current utility, depend on its beliefs about future government policy. This generates time inconsistency with a formal structure very similar to the one considered here. An example that Kydland and Prescott give is flood insurance. Once a flood has occurred, it is probably optimal for the government to transfer resources to the flood victims. But if people expect government assistance in the event of a flood, they will construct homes on flood plains and take other risky actions. So the optimal long-term policy is for the government to not offer flood insurance, as that will encourage prudent behavior. The trouble is that this "hard line" towards floods is not time-consistent. Even if the government claims it will not aid flood victims, once a flood has occurred the government will be inclined to provide aid.

You might recall time consistency from game theory, where it is known as subgame perfection. It is also worth noting that time inconsistency is a function of the return structure, not of uncertainty; an analogous time-inconsistency result holds under perfect foresight.

1.3 Recursive Formulation

To utilize dynamic programming under uncertainty requires that we impose some structure on the stochastic processes that drive our problem. Let's return to the time-consistent problem described by (PR1). Now suppose that the exogenous process $\{\eta_t\}$ is Markov in

that its conditional distribution satisfies

$$F(\eta_t | \eta_{t-1}, \eta_{t-2}, \dots) = F(\eta_t | \eta_{t-1}).$$

If we assume further that our agent observes no other variables that are useful in predicting η_t , we can construct value functions:

$$\begin{aligned} V_3(k_3, \eta_3) &= W(k_3, \eta_3), \\ V_2(k_2, \eta_2) &= \max_{x_2} U(x_2, k_2, \eta_2) + \beta E(V_3(G(x_2, k_2, \eta_3), \eta_3) | \eta_2), \\ V_1(k_1, \eta_1) &= \max_{x_1} R(x_1, k_1, \eta_1) + \beta E(V_2(G(x_1, k_1, \eta_2), \eta_2) | \eta_1), \end{aligned}$$

with expectations taken over next period's value of η . Note that η_t is important not only in that it affects returns at time t , but in that it helps predict η_{t+1} . This means that if there are other variables besides η_t that our agents use in predicting η_{t+1} , they must be included as state variables. The Markov assumption is critical: if the number of variables needed for prediction were unbounded, the number of state variables in our value functions would be unbounded as well. The Markov assumption is also not as restrictive as it might first appear. For example, if η_t follows an AR(2) process:

$$\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + \varepsilon_t,$$

with ε_t zero-mean and i.i.d., the vector (η_t, η_{t-1}) is Markov. It turns out that any finite-order ARMA process with i.i.d. innovations can be written as Markov vector, using a “state-space” representation.

If the flow return functions ($U(\cdot)$ s and $R(\cdot)$ s), transition equations ($G(\cdot)$ s) and conditional distributions ($F(\eta_t | \eta_{t-1})$) are time-invariant, then under regularity conditions, the solution to the infinite-horizon version of (PR1) involves a stochastic version of Bellman's equation:

$$V(k_t, \eta_t) = \max_{x_t} U(x_t, k_t, \eta_t) + \beta E(V(G(x_t, k_t, \eta_{t+1}), \eta_{t+1}) | \eta_t).$$

Note that the function $V(\cdot)$ is no longer indexed by t . This means that our agent's contingency plans are time-invariant as well.

2 Lag Operators and Univariate Linear Expectational Difference Equations

Many macroeconomic models involve analyzing equations of the form

$$E_t(x_{t+1}) = \lambda x_t + z_t, \quad \forall t \in T, \quad (\text{LEDE0})$$

where: $\{x_t\}$ is the stochastic process we are solving for; $E_t(\cdot)$ yields expectations conditional on time- t information; and $\{z_t\}$ is a **forcing process**, with z_t known at time t . T is the time period over which equation (LEDE0) holds; we will assume that T is either the set of all integers, so that “time” begins at $-\infty$, or the set of non-negative integers, so that time begins at 0. While the properties of these linear expectational difference equations (LEDEs) are very similar to the properties of their deterministic counterparts, their solution space is of a much higher dimension. In particular, deterministic difference equations generate sequences that are pinned down by a single initial value. Expectational difference equations, on the other hand, generate stochastic processes that must be pinned down period-by-period.

We will proceed in three steps. First, we will briefly review lag operators. Then we will turn to homogeneous LEDEs, where the forcing term z_t is identically zero. This will provide background for the final step, where we consider general LEDEs.

Those seeking more background should review Hamilton (1994), who provides a thorough treatment of lag operators and LEDEs. Sargent also covers the topic well in his book *Macroeconomic Theory* (1987). Farmer (1999) also provides a good exposition, but it omits lag operators. Technical background can be found in Gourieroux, Laffont and Monfort (1982) and Broze, Gourieroux and Szafarz (1985).

2.1 Lag Operators

Let’s start with a brief review of lag operators. Let \mathbb{S} be a set of stochastic processes, with $\{X_t\}$ a “typical” process. Define the lag operator $L^n : \mathbb{S} \rightarrow \mathbb{S}$, n an integer, by

$$L^n \{X_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty}.$$

Note that the lag operator is linear:

$$L(aX_t + bL^n X_t) = (aL + bL^{n+1})X_t,$$

so that lag operations can be manipulated like polynomials. In particular,

$$\begin{aligned}
\left(\sum_{j=0}^J (\phi L^n)^j\right) X_t &= \sum_{j=0}^J \phi^j X_{t-nj} \\
&= \left[1 + \sum_{j=1}^J (\phi L^n)^j\right] X_t \\
&= \left[1 + \phi L^n \sum_{j=0}^{J-1} (\phi L^n)^j\right] X_t \\
&= \left[1 + \phi L^n \left(-(\phi L^n)^J + \sum_{j=0}^J (\phi L^n)^j\right)\right] X_t, \\
\left[(1 - \phi L^n) \left(\sum_{j=0}^J (\phi L^n)^j\right)\right] X_t &= \left[1 - (\phi L^n)^{J+1}\right] X_t, \\
\left(\sum_{j=0}^J (\phi L^n)^j\right) X_t &= \frac{1 - (\phi L^n)^{J+1}}{1 - \phi L^n} X_t.
\end{aligned}$$

If $(\phi L^n)^{J+1} X_t$ “converges” in some relevant sense—which can happen even if $|\phi| > 1$ —we get

$$\frac{1}{1 - \phi L^n} X_t = \left(\sum_{j=0}^{\infty} (\phi L^n)^j\right) X_t.$$

What we have done is define an “inverse” of the lag operation $(1 - \phi L^n)$, a result that we will use repeatedly below.

Another useful feature of lag operators is that the lag of constant is the same constant. To see this, suppose that $X_t = c, \forall t$. Then

$$L^n c = L^n X_t = c.$$

In working with conditional expectations, we will assume that the lag operator does not shift information sets:

$$L^n E_t(X_{t+j}) = E_t(X_{t+j-n}) \neq E_{t-n}(X_{t+j-n}).$$

Making this assumption requires us to proceed with some care. Consider the processes $\{w_t\}$ and $\{y_t\}$, where w_t and y_t are both elements of the time- t information set. While the following inference is valid:

$$w_t = y_t \Leftrightarrow E_t(w_t) = E_t(y_t) \Rightarrow L^{-1}E_t(w_t) = L^{-1}E_t(y_t) \Rightarrow E_t(w_{t+1}) = E_t(y_{t+1}),$$

its converse is not:

$$E_t(w_{t+1}) = E_t(y_{t+1}) \not\Rightarrow LE_t(w_{t+1}) = LE_t(y_{t+1}) \Rightarrow w_t = y_t.$$

On the other hand:

$$w_t = y_t \Leftrightarrow E_t(w_t) = E_t(y_t) \Rightarrow LE_t(w_t) = LE_t(y_t) \Leftrightarrow w_{t-1} = y_{t-1}.$$

As a general rule, we can safely perform lag operations on conditional expectations when we either: (1) lag forward (for example, $E_t(w_{t+1}) = E_t(y_{t+1})$ implies that $E_t(w_{t+2}) = E_t(y_{t+2})$); or (2) take lags of variables that are in the original information set (for example, $w_t = y_t$ implies that $w_{t-1} = y_{t-1}$).

2.2 Homogeneous First-order LEDEs

Homogeneous LEDEs take the form

$$E_t(b_{t+1}) = \lambda b_t,$$

which can be written with lag operator notation as

$$E_t((1 - \lambda L)b_{t+1}) = 0.$$

It proves useful to convert this equation in expectations into an equation in realized values and forecast errors:

$$b_{t+1} = \lambda b_t + \varepsilon_{t+1},$$

where

$$\varepsilon_{t+1} \equiv b_{t+1} - E_t(b_{t+1}),$$

is the error of the time- t forecast of b_{t+1} . As a forecast error, ε_t forms a martingale difference sequence, which is to say that

$$E_t(\varepsilon_{t+1}) = 0.$$

We can make our problem just a bit more general by subtracting the constant $c\lambda^{t+1}$ from each side of our difference equation:

$$b_{t+1} - c\lambda^{t+1} = \lambda b_t - \lambda c\lambda^t + \varepsilon_{t+1},$$

which can be written with lag operator notation as

$$(1 - \lambda L)(b_{t+1} - c\lambda^{t+1}) = \varepsilon_{t+1}. \tag{LEDE1}$$

Our job is to not only find the constant c , as we would when solving a deterministic difference equation, but to also find the forecast error process $\{\varepsilon_t\}$.

When analyzing general LEDEs we will consider both “backward” and “forward” solutions. In the homogeneous case, however, it is straightforward to treat the forward solution as a special case of the backward solution, and so we consider just the latter. If time starts at $-\infty$, we can get the backward solution—if it is well-defined—by inverting the lag operation in (LEDE1):

$$b_{t+1} - c\lambda^{t+1} = \frac{1}{1 - \lambda L}\varepsilon_{t+1},$$

to get

$$b_t = c\lambda^t + \sum_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}.$$

If time starts at 0, we can recursively substitute to get

$$\begin{aligned} b_t &= \lambda b_{t-1} + \varepsilon_t \\ &= \lambda(\lambda b_{t-1} + \varepsilon_{t-1}) + \varepsilon_t = \lambda^2 b_{t-1} + \varepsilon_t + \lambda \varepsilon_{t-1}, \\ &\vdots \\ &= b_0 \lambda^t + \sum_{j=0}^{t-1} \lambda^j \varepsilon_{t-j}. \end{aligned}$$

The two versions of the backward solution are equivalent when

$$b_0 = c + \sum_{j=0}^{\infty} \lambda^j \varepsilon_{-j},$$

but in general b_0 can be any random variable, so that the difference equation might hold only for $t \geq 0$.

At this point, we have simply rewritten our difference equation, and done nothing to narrow the set of permissible solutions. One way to narrow the solution set would be to directly impose $\{\varepsilon_t\}_{t=1}^{\infty}$ and b_0 . For example, the assumption of perfect foresight requires that $\varepsilon_t = 0, \forall t$. We will often refer to this sort of restriction as an “initial condition,” but technically we are imposing a forecast error at every point in time.

A second restriction is to require the process for $\{b_t\}$ to be “bounded” in some sense. The exact form of the boundedness restriction depends on the norm (measure of size) in effect, as well as the underlying economic model, which has the unfortunate implication that not all boundedness restrictions are equivalent. A common and useful set of

restrictions, though, is to require that the conditional mean not explode:

$$\lim_{J \rightarrow \infty} E_t(b_{t+J}) = 0, \quad \forall t,$$

and that the unconditional variance be bounded:

$$\sup_t V(b_t) < \infty.$$

The first restriction is especially useful, because we can use the law of iterated expectations to show that

$$E_t(b_{t+2}) = E_t(E_{t+1}(b_{t+2})) = E_t(\lambda b_{t+1}) = \lambda(\lambda b_t),$$

so that

$$E_t(b_{t+j}) = \lambda^j b_t.$$

Then if $|\lambda| \geq 1$, the only admissible solution is $\varepsilon_t = c = b_0 = 0$, so that $b_t = 0, \forall t$. It can be shown, using the approach below, that this is also the forward solution to (LEDE1). But if $|\lambda| < 1$, there are many c and $\{\varepsilon_t\}$ that satisfy our boundedness conditions; for example, any ε_t that follows a zero-mean i.i.d. process will work.

Finally, let's consider the two sets of restrictions together. First note that if $|\lambda| \geq 1$, b_t cannot satisfy both an “initial condition” and a non-explosiveness condition. To not explode b_t must equal zero. But this means that b_0 and ε_t must equal zero as well—they cannot be given arbitrary values. On the other hand, if $|\lambda| < 1$, the boundedness condition will not be enough to yield a unique solution—we will need an initial condition as well.

2.3 General LEDEs

2.3.1 The Problem

Finally, we return to the main problem,

$$E_t(x_{t+1}) = \lambda x_t + z_t, \tag{LEDE0}$$

where $\{z_t\}$ is a stochastic forcing process. We will assume that $\{z_t\}$ is “well-behaved” in that if ϕ is of modulus (generalized absolute value) less than one:

$$\begin{aligned} \lim_{K \rightarrow \infty} E_t \left(\sum_{j=0}^{\infty} \phi^j z_{t+K+j} \right) &= 0, \\ \sup_t V \left(\sum_{j=0}^{\infty} \phi^j z_{t-j} \right) &< \infty, \end{aligned}$$

By way of example, $\{z_t\}$ could be a stationary ARMA process with i.i.d. innovations.

Proceeding like before, we can make our problem more general by subtracting a “bubble term” from each side of (LEDE0):

$$E_t(x_{t+1} - b_{t+1}) = \lambda x_t + z_t - \lambda b_t,$$

with b_t satisfying

$$E_t(b_{t+1}) = \lambda b_t.$$

This can be written with lag operator notation as

$$E_t((1 - \lambda L)(x_{t+1} - b_{t+1})) = z_t.$$

It proves useful to convert this equation in expectations into an equation in realized values and forecast errors:

$$\begin{aligned} x_{t+1} - b_{t+1} &= \lambda(x_t - b_t) + \tilde{\eta}_{t+1} + z_t, \\ \tilde{\eta}_{t+1} &\equiv (x_{t+1} - b_{t+1}) - E_t(x_{t+1} - b_{t+1}), \end{aligned}$$

which can be written with lag operator notation as:

$$(1 - \lambda L)(x_{t+1} - b_{t+1}) = \tilde{\eta}_{t+1} + z_t. \tag{LEDE2}$$

Note that $\tilde{\eta}_t$ is the forecast error not for x_t but for the “fundamental” part of x_t : $x_t - b_t$. Our goal is to find the forecast error process $\{\tilde{\eta}_t\}$ and the bubble process $\{b_t\}$.

2.3.2 Backward and Forward Solutions

We begin with the backward solution. If time starts at $-\infty$, we can get the backward solution—if it is well-defined—by inverting the lag operation in (LEDE2):

$$\begin{aligned} x_{t+1} - b_{t+1} &= \frac{1}{1 - \lambda L} (\tilde{\eta}_{t+1} + z_t), \\ x_{t+1} &= \sum_{j=0}^{\infty} \lambda^j (z_{t-j} + \tilde{\eta}_{t+1-j}) + b_{t+1}. \end{aligned}$$

Careful inspection shows, however, that $\{\tilde{\eta}_t\}$ and $\{b_t\}$ cannot be identified separately. In particular, we can rewrite our solution as

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \tilde{b}_{t+1}, \\ \tilde{b}_{t+1} &\equiv b_{t+1} + \sum_{j=0}^{\infty} \lambda^j \tilde{\eta}_{t+1-j}, \end{aligned}$$

and it is straightforward to show that \tilde{b}_{t+1} is also a bubble term. For this reason, $\sum_{j=0}^{\infty} \lambda^j z_{t-j}$ is sometimes called the fundamental component of the backward solution.

If time starts at 0, we can recursively substitute to get

$$x_{t+1} = \sum_{j=0}^t \lambda^j z_{t-j} + \sum_{j=0}^t \lambda^j \tilde{\eta}_{t+1-j} + (x_0 - b_0) \lambda^{t+1} + b_{t+1},$$

which after a few manipulations can be expressed as

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^t \lambda^j z_{t-j} + \sum_{j=0}^t \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}, \\ \eta_t &\equiv \tilde{\eta}_t + (b_t - E_{t-1}(b_t)) = x_t - E_{t-1}(x_t). \end{aligned}$$

The two versions of the backward solution are equivalent when

$$x_0 = \tilde{b}_0 + \sum_{j=0}^{\infty} \lambda^j z_{-j-1},$$

but in general x_0 can be any random variable, so that the difference equation might hold only for $t \geq 0$. Since η_t is the forecast error for x_t , the time-0 formulation is in many ways the more useful one. x_t equals the sum of its fundamental solution, its discounted initial value, and the discounted sum of its forecast errors, and it is natural to think of narrowing the solution set by picking an initial value and a process for the forecast errors.

Now consider the forward solution. First, rewrite (LEDE2) as:

$$\begin{aligned} \left(\frac{1 - \lambda L}{-\lambda L} \right) (-\lambda L) (x_{t+1} - b_{t+1}) &= \tilde{\eta}_{t+1} + z_t, \\ (1 - \lambda^{-1} L^{-1}) (x_t - b_t) &= -\frac{1}{\lambda} (z_t + \tilde{\eta}_{t+1}). \end{aligned}$$

Next, we require x_t to be a function only of variables known at time t :

$$E_t \left((1 - \lambda^{-1} L^{-1}) (x_t - b_t) \right) = -\frac{1}{\lambda} E_t (z_t + \tilde{\eta}_{t+1}).$$

Invert $(1 - \lambda^{-1} L^{-1})$ to get

$$\begin{aligned} E_t (x_t - b_t) &= -\frac{1}{\lambda} E_t \left(\frac{1}{1 - \lambda^{-1} L^{-1}} (z_t + \tilde{\eta}_{t+1}) \right), \\ x_t &= -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j (z_{t+j} + \tilde{\eta}_{t+j+1}) \right) + b_t \\ &= -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) + b_t. \end{aligned}$$

because $E_t (\tilde{\eta}_{t+j}) = 0, \forall j \geq 1$. $-\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right)$ —if it is well-defined—is sometimes called the fundamental component of the forward solution. Note that the forecast error for the fundamental component is

$$\tilde{\eta}_t = -\frac{1}{\lambda} \left[E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) - E_{t-1} \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) \right].$$

2.3.3 Restrictions

In the previous section, we derived a forward solution:

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right) + b_t,$$

and two versions of the backward solution:

$$\begin{aligned} x_t &= \sum_{j=0}^{\infty} \lambda^j z_{t-1-j} + b_t, \\ x_t &= \sum_{j=0}^{t-1} \lambda^j z_{t-1-j} + \sum_{j=0}^{t-1} \lambda^j \eta_{t-j} + x_0 \lambda^t. \end{aligned}$$

At this point, we have simply rewritten our difference equation, and done nothing to narrow the set of permissible solutions. One way to narrow the solution set would be to directly impose $\{\eta_t\}_{t=1}^{\infty}$ and x_0 . For example, a standard capital accumulation equation is

$$k_{t+1} = (1 - \delta)k_t + i_t, \quad k_0 \text{ given}, \quad k_{t+1} - E_t(k_{t+1}) = 0.$$

We will often refer to this sort of restriction as an “initial condition.”

A second restriction is to require the process for $\{x_t\}$ to be “bounded” in some sense. One interpretation is to require that the conditional mean not explode:

$$\lim_{j \rightarrow \infty} E_t(x_{t+j}) = 0, \quad \forall t,$$

and that the unconditional variance be bounded:

$$\sup_t V(x_t) < \infty.$$

If $|\lambda| < 1$, we can invert $1 - \lambda L$ backwards to get

$$x_t = \sum_{j=0}^{\infty} \lambda^j z_{t-1-j} + b_t,$$

where b_t is a bubble term. Since we have restricted $\{z_t\}$ to be “well-behaved,” this is a well-defined sum, and, recalling section 2.2, there are a large number of permissible $\{b_t\}$. Alternatively, we can recursively substitute to get

$$x_t = \sum_{j=0}^{t-1} \lambda^j z_{t-1-j} + \sum_{j=0}^{t-1} \lambda^j \eta_{t-j} + x_0 \lambda^t,$$

with a large number of permissible $\{\eta_t\}$ and x_0 .

Recall from the discussion of homogeneous LEDEs that when $|\lambda| > 1$, b_t will explode unless it is identically zero. Similarly, when $|\lambda| > 1$ in the general case, we will invert $1 - \lambda L$ forward and set $b_t = 0$, so that

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right).$$

Since we have restricted $\{z_t\}$ to be “well-behaved,” this is a well-defined sum.

2.4 Rule of Thumb

Considering both sets of restrictions yields the following rule of thumb. When $|\lambda| < 1$, solve backward to get

$$x_t = \sum_{j=0}^{t-1} \lambda^j z_{t-1-j} + \sum_{j=0}^{t-1} \lambda^j \eta_{t-j} + x_0 \lambda^t,$$

and use “initial conditions” to pin down x_0 and $\{\eta_t\}$. When $|\lambda| > 1$, solve forward and set $b_t = 0$ to get

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right).$$

The razor’s edge case, $|\lambda| = 1$, requires case-by-case analysis.

Note that when $|\lambda| > 1$, x_t cannot satisfy both an “initial condition” and a non-explosiveness condition. To not explode, x_t must equal its fundamental forward solution. This immediately restricts x_0 and η_t as well—they cannot be given arbitrary values. On the other hand, if $|\lambda| < 1$, the boundedness condition will not be enough to yield a unique solution—we will need an initial condition as well. What this suggests is that the real task lies not in solving difference equations *per se*, but rather in interpreting the set of solutions that is consistent with the restrictions—non-explosiveness and initial conditions—implied by economic theory.

3 Saddle-Path Stability in Vector Linear Expectational Difference Equations

In the previous section, we analyzed univariate linear expectational difference equations (LEDEs) of the form

$$E_t(x_{t+1}) = \lambda x_t + z_t.$$

Once one is comfortable with them, finding “non-explosive” solutions for univariate LEDEs is usually straightforward. If the root of the equation (λ) has a modulus (the absolute value of a complex number) greater than one, solve the equation forward. x_t then becomes a weighted sum of expected future values of the forcing variable z_t . If the root of the equation has a modulus less than one, solve the equation backward. x_t then becomes a weighted sum of its initial value x_0 , current and lagged values of the forecast error η_t and current and lagged values of the forcing variable z_t . One then employs some sort of external information to pin down x_0 and $\{\eta_t\}$.

The main problem occurs when the restrictions imposed by economic theory—such as non-explosiveness conditions or initial conditions for capital stocks—do not mesh well with the solution dictated by the size of λ . In particular, when $|\lambda| > 1$, the forward solution that satisfies general non-explosiveness conditions is unique, and cannot be modified to satisfy arbitrary initial conditions as well. And when $|\lambda| < 1$, in the absence of an initial condition there will be an infinity of processes that satisfy non-explosiveness conditions. (It is important to note, however, that some analysts consider a multiplicity of solutions to be a desirable result.)

So univariate LEDEs are straightforward to solve; our concern is whether the set of solutions that satisfy the economic restrictions is the “right” size. Vector LEDEs are a different matter. Not only is it possible that the structure of the system generates a solution set of the wrong size, but it is much more difficult to determine what the size of this solution set is.

We are seeking processes that solve the following vector LEDE:

$$E_t(\tilde{\mathbf{y}}_{t+1}) = \tilde{\mathbf{B}}(\tilde{\mathbf{y}}_t) + \mathbf{\Gamma}\mathbf{z}_t,$$

where $\tilde{\mathbf{y}}_t$ is the random vector of interest, and \mathbf{z}_t is an exogenous forcing vector that follows

$$\mathbf{z}_{t+1} = \mathbf{\Omega}\mathbf{z}_t + \boldsymbol{\varepsilon}_{t+1}, \quad E_t(\boldsymbol{\varepsilon}_{t+1}) = \mathbf{0}, \quad \mathbf{z}_0 \text{ given.}$$

Since any ARMA process can be written as an AR(1) process by using a state-space representation, this not so restrictive an assumption. $\tilde{\mathbf{B}}$, $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ are real matrices. We can combine these two equations to get

$$E_t \begin{pmatrix} \tilde{\mathbf{y}}_{t+1} \\ \mathbf{z}_{t+1} \end{pmatrix} = \begin{bmatrix} \tilde{\mathbf{B}} & \mathbf{\Gamma} \\ \mathbf{0} & \mathbf{\Omega} \end{bmatrix} \begin{pmatrix} \tilde{\mathbf{y}}_t \\ \mathbf{z}_t \end{pmatrix}.$$

Relabelling, we get our benchmark equation:

$$E_t(\mathbf{y}_{t+1}) = \mathbf{B}(\mathbf{y}_t). \tag{LEDE0}$$

which is accompanied by the side condition

$$\mathbf{z}_{t+1} - E_t(\mathbf{z}_{t+1}) = \boldsymbol{\varepsilon}_{t+1}, \quad \mathbf{z}_0 \text{ given},$$

and, possibly, similar restrictions on some of the elements of $\tilde{\mathbf{y}}$. Our goal is to find stochastic processes $\{\mathbf{y}_t\}$ that satisfy (LEDE0), the side restrictions, and a non-explosiveness condition.

There are several general approaches for solving vector LEDEs. The approach that we will take consists of three steps:

1. Transform the original “coupled” system, given by (LEDE0), into an “uncoupled system” where each transformed variable depends only on its own lagged value.
2. Solve the transformed system by imposing restrictions that apply to the original coupled system.
3. Recover the original system by “reverse-transforming” the solution to the transformed system.

The exposition below also proceeds in three steps. First, we consider the solution to an uncoupled system. Since each variable of an uncoupled system is independent of the others, this reduces to solving a sequence of univariate LEDEs. Next, we consider how one solves for random vectors that are linear combinations of the variables in an uncoupled system. Finally, we show how the general system in equation (LEDE0) can be transformed into an uncoupled system. To keep the algebra simple, we will work with the two-variable case, and briefly discuss how to generalize to bigger systems.

By way of background, the approach that we will use was introduced to economics by Blanchard and Kahn (1980). A generalization of this approach, along with a nice

exposition, can be found in Klein (1998). Additional results appear in Broze, Gourieroux and Szafarz (1985, 1995). The best introduction to the topic probably appears in Farmer’s *Macroeconomics of Self-fulfilling Prophecies* (1999). Our approach is also an extension of the standard approach for solving linear difference equations under perfect foresight, and thus appears, with the requisite modifications, in many textbooks. A particularly clear exposition appears in Azariadis’ (1993) book.

3.1 An Uncoupled System

Consider the following uncoupled vector LEDE:

$$E_t \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} = \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} \equiv \Phi \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix}, \quad (\text{LEDE1})$$

$$|\phi_1| > 1, \quad |\phi_2| < 1.$$

Note that this system is nothing more than a stacked pair of univariate LEDEs. As one might expect, the configuration of the “roots” of Φ will turn out to be essential. Recalling our analysis of univariate systems in section 2, the “non-explosive” solutions to (LEDE1) are

$$\begin{aligned} x_t^1 &= 0, \\ x_t^2 &= \phi_2^t x_0^2 + \sum_{j=0}^{t-1} \phi_2^j \varepsilon_{t-j}^2, \\ E_{t-1}(\varepsilon_t^2) &= 0, \end{aligned}$$

which can be written as

$$\begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} = \Phi \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_{t+1}^2 \end{pmatrix}, \quad x_0^1 = 0. \quad (\text{LEDE1}')$$

In the next section, we consider some restrictions that will pin down the initial value x_0^2 and forecast errors $\{\varepsilon_t^2\}$.

3.2 Coupled Systems: Part 1

Now consider a linear transformation of the random vector in equation (LEDE0):

$$\begin{aligned} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} \\ &\equiv \mathbf{A} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix}, \end{aligned}$$

where \mathbf{A} is non-singular, and $a_{22} \neq 0$.

Suppose that we wanted to find the set of processes $\{\mathbf{y}_t\} = \{y_t^1, y_t^2\}$ that satisfy: (1) y_t^1 and y_t^2 do not explode; (2) y_0^2 and $\eta_{t+1}^2 \equiv y_{t+1}^2 - E_t(y_{t+1}^2)$ are given. We will sometimes refer to the second set of restrictions as “initial conditions.” Since $\{\mathbf{x}_t\}$ has a nice uncoupled structure, it seems sensible to first solve for x_t^1 and x_t^2 , use the restrictions on y_t^1 and y_t^2 to narrow the set of solutions, and then finally to back out y_t^1 and y_t^2 using $\mathbf{y}_t = \mathbf{A}\mathbf{x}_t$.

First, note that y_t^1 and y_t^2 do not explode if and only if x_t^1 and x_t^2 do not explode. This immediately implies that $x_t^1 = 0$, so that

$$\begin{aligned} y_t^2 &= a_{21}x_t^1 + a_{22}x_t^2 = a_{22}x_t^2 \\ \Leftrightarrow x_t^2 &= \frac{1}{a_{22}}y_t^2. \end{aligned}$$

Using this result, we impose the second restriction as follows:

$$\begin{aligned} x_0^2 &= \frac{1}{a_{22}}y_0^2, \\ \varepsilon_{t+1}^2 &= \frac{1}{a_{22}}\eta_{t+1}^2. \end{aligned}$$

The restrictions on y_0^2 and $\{\eta_t^2\}$ thus pin down x_0^2 and $\{\varepsilon_t^2\}$, so that the process for x_t^2 is unique as well.

Using these results immediately delivers y_t^1 :

$$\begin{aligned}
y_t^1 &= a_{11}x_t^1 + a_{12}x_t^2 \\
&= a_{12}x_t^2 \\
&= a_{12}\frac{1}{a_{22}}y_t^2, \\
y_0^1 &= \frac{a_{12}}{a_{22}}y_0^2, \\
\eta_{t+1}^1 &\equiv y_{t+1}^1 - E_t(y_{t+1}^1) \\
&= \frac{a_{12}}{a_{22}}\eta_{t+1}^2.
\end{aligned}$$

Collecting the preceding results, we can write our system as

$$\begin{aligned}
\begin{pmatrix} y_{t+1}^1 \\ y_{t+1}^2 \end{pmatrix} &= \mathbf{A} \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} = \mathbf{A} \left[\Phi \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_{t+1}^2 \end{pmatrix} \right] \\
&= \mathbf{A} \left[\Phi \begin{pmatrix} 0 \\ y_t^2/a_{22} \end{pmatrix} + \begin{pmatrix} 0 \\ \eta_{t+1}^2/a_{22} \end{pmatrix} \right] \\
&= \begin{bmatrix} a_{11}\phi_1 & a_{12}\phi_2 \\ a_{21}\phi_1 & a_{22}\phi_2 \end{bmatrix} \begin{pmatrix} 0 \\ y_t^2/a_{22} \end{pmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} 0 \\ \eta_{t+1}^2/a_{22} \end{pmatrix} \\
&= \begin{bmatrix} 0 & (a_{12}/a_{22})\phi_2 \\ 0 & \phi_2 \end{bmatrix} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} + \begin{pmatrix} a_{12}/a_{22} \\ 1 \end{pmatrix} \eta_{t+1}^2, \\
y_0^1 &= \frac{a_{12}}{a_{22}}y_0^2,
\end{aligned}$$

Another way to write this solution is to modify (LEDE1') and require that $\varepsilon_{t+1}^2 = (1/a_{22})\eta_{t+1}^2$:

$$\begin{aligned}
\begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} &= \Phi \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_{t+1}^2 \end{pmatrix}, \\
\mathbf{A} \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} &= \mathbf{A}\Phi\mathbf{A}^{-1}\mathbf{A} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} + \mathbf{A} \begin{pmatrix} 0 \\ (1/a_{22})\eta_{t+1}^2 \end{pmatrix}, \\
\begin{pmatrix} y_{t+1}^1 \\ y_{t+1}^2 \end{pmatrix} &= \mathbf{B} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} + \begin{pmatrix} a_{12}/a_{22} \\ 1 \end{pmatrix} \eta_{t+1}^2, \\
\mathbf{B} &\equiv \mathbf{A}\Phi\mathbf{A}^{-1}
\end{aligned}$$

As long as $y_0^1 = (a_{12}/a_{22})y_0^2$, a bit of algebra will show that the two equations for \mathbf{y}_{t+1} are

equivalent. In particular, both formulations reveal a stochastic singularity: y_t^1 is an exact linear function of y_t^2 , so that the forecast error for y_t^1 is a linear function of the forecast error for y_t^2 .

The system we have just solved is said to be saddle-path stable, in that there is a unique process for $\{\mathbf{y}_t\}$ that satisfies both the non-explosiveness condition and the side conditions on $\{y_t^2\}$. Note that the key to saddle-path stability is the configuration of Φ ; since $\{\mathbf{y}_t\}$ is a non-singular transformation of $\{\mathbf{x}_t\}$, the solution for $\{\mathbf{y}_t\}$ will be unique if and only if the solution for $\{\mathbf{x}_t\}$ is unique, and this depends on the elements of Φ . In particular, $|\phi_1| > 1$ means that the only value of $\{x_t^1\}$ that will not explode is $\{0\}$, while $|\phi_2| < 1$ means that the set of non-exploding processes for $\{x_t^2\}$ is rich enough to satisfy the side conditions on $\{y_t^2\}$.

Let's briefly consider what would happen if Φ had a different configuration. If both ϕ_1 and ϕ_2 were of modulus greater than 1, x_t^1 and x_t^2 would both explode unless both variables were identically zero. This means that $\{x_t^2\}$ could not be set to satisfy arbitrary values of y_0^2 and $\{\eta_{t+1}^2\}$, which in turn implies that $\{y_t^1, y_t^2\}$ could not satisfy both non-explosiveness conditions and the side restrictions on $\{y_t^2\}$. On the other hand, if both ϕ_1 and ϕ_2 were both of modulus less than 1, there would be a large number of processes for $\{x_t^2\}$ and $\{x_t^1\}$ that would be stable. Even if we used the initial conditions on $\{y_t^2\}$ to pin down $\{x_t^2\}$, $\{x_t^1\}$ would be unrestricted. This means that $\{y_t^1, y_t^2\}$ would be unique only if we had *two* sets of initial conditions.

3.3 Coupled Systems, Part 2

In the previous section, we assumed that the random vector \mathbf{y}_t could be written as a linear transformation of the random vector \mathbf{x}_t , which had a clean uncoupled structure. In this section, we show that such a transformation usually exists. To begin, consider a generic coupled system:

$$E_t \begin{pmatrix} y_{t+1}^1 \\ y_{t+1}^2 \end{pmatrix} = \mathbf{B} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix}, \quad (\text{LEDE2})$$

where \mathbf{B} is not necessarily diagonal. Our goal is to write \mathbf{y}_t as $\mathbf{A}\mathbf{x}_t$, with the elements of \mathbf{x}_t uncoupled like the variables in equation (LEDE1).

To do this we will utilize eigenvalues and eigenvectors. Recall that when

$$\mathbf{B}\mathbf{a} = \phi\mathbf{a},$$

\mathbf{a} is said to an eigenvector of \mathbf{B} , and ϕ is the associated eigenvalue. More background

can be found in any decent linear algebra textbook, as well as in economic texts such as Azariadis' (1993).

Since \mathbf{B} is a 2×2 matrix, it has two (not necessarily distinct) eigenvalues, ϕ_1 and ϕ_2 , and two associated eigenvectors \mathbf{a}_1 and \mathbf{a}_2 , so that

$$\begin{aligned}\mathbf{B} \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} &= \begin{bmatrix} \phi_1 \mathbf{a}_1 & \phi_2 \mathbf{a}_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix},\end{aligned}$$

which we can write as

$$\begin{aligned}\mathbf{B}\mathbf{A} &= \mathbf{A}\mathbf{\Phi}, \\ \mathbf{A} &\equiv \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}, \\ \mathbf{\Phi} &\equiv \begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}.\end{aligned}$$

Moreover, when \mathbf{A} is invertible:

$$\mathbf{B} = \mathbf{B}\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}\mathbf{\Phi}\mathbf{A}^{-1}.$$

What we have done is to diagonalize \mathbf{B} .

Next, define

$$\begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix},$$

and rewrite equation (LEDE2) as

$$\begin{aligned}E_t \mathbf{A}\mathbf{A}^{-1} \begin{pmatrix} y_{t+1}^1 \\ y_{t+1}^2 \end{pmatrix} &= \mathbf{A}\mathbf{\Phi}\mathbf{A}^{-1} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix}, \\ E_t \mathbf{A} \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} &= \mathbf{A}\mathbf{\Phi} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix}, \\ E_t \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \end{pmatrix} &= \mathbf{\Phi} \begin{pmatrix} x_t^1 \\ x_t^2 \end{pmatrix},\end{aligned}$$

which, since $\mathbf{\Phi}$ is diagonal, is a version of (LEDE1)!

Now we have our solution strategy:

1. Diagonalize \mathbf{B} , and define $\mathbf{x}_t = \mathbf{A}^{-1}\mathbf{y}_t$.
2. Solve for the elements of \mathbf{x}_t , using restrictions that apply to the coupled system, as described in section 3.2.
3. “Reverse-transform” to find $\mathbf{y}_t = \mathbf{A}\mathbf{x}_t$, so that

$$\begin{pmatrix} y_{t+1}^1 \\ y_{t+1}^2 \end{pmatrix} = \mathbf{B} \begin{pmatrix} y_t^1 \\ y_t^2 \end{pmatrix} + \begin{pmatrix} a_{12}/a_{22} \\ 1 \end{pmatrix} \eta_{t+1}^2, \quad y_0^1 = \frac{a_{12}}{a_{22}} y_0^2.$$

3.4 Rule of Thumb

It is straightforward to generalize these results to larger systems. Suppose that the process $\{\mathbf{y}_t\}$ in equation (LEDE0) is subject to n sets of initial conditions and a general non-explosiveness condition. Suppose further that the matrix \mathbf{B} has m eigenvalues with modulus less than one. If $m < n$, equation (LEDE0) will have no non-explosive solutions that also satisfy all of the initial conditions. If $m > n$, there will be an infinity of solutions; in a sense we have “too few” initial conditions. Finally if $m = n$, there will be exactly one process $\{\mathbf{y}_t\}$ that jointly satisfies the initial and the non-explosiveness conditions. Our results also generalize to cases where \mathbf{B} is not diagonalizable. In those cases, there are alternate, more complex, decompositions that work just like diagonalization. In particular, even when \mathbf{B} is not diagonalizable, we can characterize the solutions to equation (LEDE0) by looking at the eigenvalues of \mathbf{B} .

This leaves two remaining issues. The first is mechanical: how do we characterize the eigenvalues of \mathbf{B} ? Unfortunately, the analytic results are fairly meager. While 2×2 matrices are fairly straightforward to characterize, larger systems drive us to numerical analysis. The second remaining issue is economic: which elements of $\{\mathbf{y}_t\}$ should be subject to initial conditions? The general rule of thumb is that “state” variables such as capital stocks or exogenous shocks should obey initial conditions, while “control” variables such as consumption, output or asset prices should not. Then when our system is saddle-path stable, the control variables are linear functions of the states. But, as Klein (1988) points out, there is always room for interpretation.

3.5 An Example

To illustrate our methodology, let's work through a consumption example. Consider an individual that maximizes the expected present value of her utility from consumption during an infinite lifetime:

$$\begin{aligned} & \max_{\{\tilde{c}_t\}_{t=0}^{\infty}} E_0 \left(\sum_{t=0}^{\infty} \tilde{\beta}^t \left[\alpha \tilde{c}_t - \frac{1}{2} \tilde{c}_t^2 \right] \right), \\ & \text{s.t. } \tilde{A}_{t+1} = R \left(\tilde{A}_t + y + \varepsilon_t - \tilde{c}_t \right), \\ & \lim_{J \rightarrow \infty} E_t \left\{ R^{-J} \tilde{A}_{t+J} \right\} = 0, \\ & \tilde{A}_0 \text{ given,} \end{aligned}$$

where: \tilde{c}_t is consumption; \tilde{A}_t is the real value of risk-free assets; $0 < y < \alpha$ is (essentially) permanent income; and ε_t is exogenous transitory income. We will assume that $\{\varepsilon_t\}$ is a stationary martingale difference sequence, so that

$$E_t(\varepsilon_{t+1}) = 0. \tag{TS}$$

While income is stochastic, we will assume that $0 < c_t < \alpha$, so that we can consider interior solutions, and we will require $\tilde{\beta}$ and R to satisfy

$$\tilde{\beta} \in (0, 1), \quad \tilde{\beta}R > 1.$$

The Euler equation for this problem is

$$\alpha - \tilde{c}_t = \tilde{\beta} E_t((\alpha - \tilde{c}_{t+1}) R).$$

While we could work with the Euler equation and the consumer's flow budget constraint, our analysis will be a lot cleaner if we can eliminate some constants. Fortunately, this simply requires a few definitions. In particular, the Euler equation can be written as

$$\begin{aligned} E_t(c_{t+1}) &= \beta^{-1} c_t, & \text{(EE)} \\ c_t &\equiv \tilde{c}_t - \alpha, \\ \beta &\equiv \tilde{\beta}R, \end{aligned}$$

and a little bit of algebra shows that the consumer's flow budget constraint can be written

as

$$\begin{aligned} A_{t+1} &= R(A_t + \varepsilon_t - c_t), & (\text{FBC}) \\ A_{t+1} &\equiv \tilde{A}_t - \frac{R}{R-1}(\alpha - y). \end{aligned}$$

For the remainder of the example, we will work with these transformed equations, along with the constraints

$$\lim_{J \rightarrow \infty} E_t \{ R^{-J} A_{t+J} \} = 0, \quad \forall t, \quad (\text{ENPG})$$

$$A_0 \text{ given, } A_{t+1} - E_t(A_{t+1}) = 0, \quad \forall t, \quad (\text{A0})$$

$$\varepsilon_0 \text{ given, } \varepsilon_{t+1} - E_t(\varepsilon_{t+1}) = \varepsilon_{t+1}, \quad \forall t. \quad (\text{E0})$$

Following our work on optimal consumption, we can find the solution to this system by recursive substitution. In particular, if we iterate on (FBC) in the forward direction and impose (ENPG), we will get the expected present value budget constraint:

$$\begin{aligned} E_t \left\{ \sum_{j=0}^{\infty} R^{-j} c_{t+j} \right\} &= E_t \left\{ A_t + \sum_{j=0}^{\infty} R^{-j} \varepsilon_{t+j} \right\} & (\text{EPVBC}) \\ &= A_t + \varepsilon_t. \end{aligned}$$

Moreover, recursive substitution with (EE) shows that

$$E_t(c_{t+j}) = \beta^{-j} c_t, \quad \forall j \geq 0,$$

which, when combined with (EPVBC) yields

$$\begin{aligned} \sum_{j=0}^{\infty} (\beta R)^{-j} c_t &= A_t + \varepsilon_t, \\ c_t &= [1 - (\beta R)^{-1}] \cdot [A_t + \varepsilon_t]. \end{aligned}$$

Let's see if we can derive the same answer using the diagonalization approach presented above. First, write equations (EE), (FBC) and (TS) as a system of the form shown in (LEDE2):

$$E_t \begin{pmatrix} c_{t+1} \\ A_{t+1} \\ \varepsilon_{t+1} \end{pmatrix} = \begin{bmatrix} \beta^{-1} & 0 & 0 \\ -R & R & R \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} c_t \\ A_t \\ \varepsilon_t \end{pmatrix} \equiv \mathbf{B} \begin{pmatrix} c_t \\ A_t \\ \varepsilon_t \end{pmatrix},$$

along with the constraints (ENPG), (A0) and (E0). The next step is to find the eigenvalues

and eigenvectors of \mathbf{B} . Each eigenvalue-eigenvector pair must satisfy

$$[\mathbf{B} - \phi_j \mathbf{I}_3] \mathbf{a}_j = \mathbf{0}, \quad j = 1, 2, 3,$$

where \mathbf{I}_3 is the identity matrix. We find $\{\phi_j\}$ by finding the values of ϕ that set the determinant $|\mathbf{B} - \phi \mathbf{I}_3|$ to zero, which boils down to finding the roots of

$$(\beta^{-1} - \phi)(R - \phi)(-\phi) = 0.$$

With the three roots $(R, \beta^{-1}, 0)$ in hand, we find that the eigenvalues and eigenvectors of \mathbf{B} are

$$\begin{aligned} \phi_1 = R, \quad \mathbf{a}_1 &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}', \\ \phi_2 = \beta^{-1}, \quad \mathbf{a}_2 &= \begin{bmatrix} (1 - (\beta R)^{-1}) & 1 & 0 \end{bmatrix}', \\ \phi_3 = 0, \quad \mathbf{a}_3 &= \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}'. \end{aligned}$$

You should take a moment to confirm that these are in fact the eigenvalues and eigenvectors of \mathbf{B} . (If you are a stickler for detail, any good linear algebra book will show how to find the eigenvectors.) Note that our system is saddle-path stable: we have two eigenvalues of modulus less than one (β^{-1} and 0) to go with the two variables, A_t and ε_t , that have to satisfy initial conditions.

We can now diagonalize \mathbf{B} :

$$\begin{aligned} \mathbf{B} &= \mathbf{A} \mathbf{\Phi} \mathbf{A}^{-1}, \\ \mathbf{A} &\equiv \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 - (\beta R)^{-1} & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \\ \mathbf{\Phi} &\equiv \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

For future reference, partition \mathbf{A} as

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix},$$

$\begin{matrix} (1 \times 1) & (1 \times 2) \\ (2 \times 1) & (2 \times 2) \end{matrix}$

These matrices/vectors correspond to the scalars a_1, a_2, a_3, a_4 used in the previous section.

Next, define the vector of transformed variables

$$\begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} c_t \\ A_t \\ \varepsilon_t \end{pmatrix} \Leftrightarrow \begin{pmatrix} c_t \\ A_t \\ \varepsilon_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{bmatrix} 0 & 1 - (\beta R)^{-1} & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix},$$

where

$$E_t \begin{pmatrix} x_{t+1}^1 \\ x_{t+1}^2 \\ x_{t+1}^3 \end{pmatrix} = \Phi \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \end{pmatrix}.$$

Note that we have ordered the eigenvalues and eigenvectors of \mathbf{B} so that the explosive eigenvalue, R , appears in the first row. Note further that A_t will satisfy (ENPG) only if $x_t^1 = 0$, which means that

$$\begin{pmatrix} c_t \\ A_t \\ \varepsilon_t \end{pmatrix} = \mathbf{A} \begin{pmatrix} 0 \\ x_t^2 \\ x_t^3 \end{pmatrix} = \begin{bmatrix} 1 - (\beta R)^{-1} & 0 \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix}. \quad (\text{TRANS})$$

Recall that A_t and ε_t must satisfy initial conditions, namely (A0) and (E0). It follows from the equation (TRANS) that

$$\begin{aligned} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix} &= \mathbf{A}_{22} \begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix} \\ &\equiv \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix}, \end{aligned}$$

with the submatrix \mathbf{A}_{22} being used in the same way as the scalar a_{22} was used in the preceding sections. With this result we can pin down x_t^2 and x_t^3 by requiring them to satisfy the initial conditions (A0) and (E0) through

$$\begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix} = \mathbf{A}_{22}^{-1} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix}.$$

Combining this result with equation (TRANS), we can express consumption, c_t , as a

function of assets, A_t , and transitory income, ε_t , by

$$\begin{aligned} c_t &= \mathbf{A}_{12} \begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix} \\ &\equiv \begin{bmatrix} 1 - (\beta R)^{-1} & 0 \end{bmatrix} \begin{pmatrix} x_t^2 \\ x_t^3 \end{pmatrix} \\ &= \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix}, \end{aligned}$$

with \mathbf{A}_{12} and \mathbf{A}_{22} corresponding to the coefficients a_{12} and a_{22} in the scalar derivations of the preceding sections. Substituting for \mathbf{A}_{12} and \mathbf{A}_{22} , we get

$$\begin{aligned} c_t &= \begin{bmatrix} 1 - (\beta R)^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix} \\ &= \begin{bmatrix} 1 - (\beta R)^{-1} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix} \\ &= [1 - (\beta R)^{-1}] \cdot [A_t + \varepsilon_t], \end{aligned}$$

which is—fortunately!—the answer we derived before. Recall that in the two-variable case, the variable that did not have to satisfy initial conditions, y_t^1 , was an exact linear function of the variable that did have to satisfy initial conditions, y_t^2 , through

$$y_t^1 = a_{12} \frac{1}{a_{22}} y_t^2.$$

In this example, y_t^1 corresponds to c_t , while y_t^2 corresponds to the vector $\begin{bmatrix} A_t & \varepsilon_t \end{bmatrix}'$. In particular, we find that

$$y_t^1 = c_t = \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \begin{pmatrix} A_t \\ \varepsilon_t \end{pmatrix} = \mathbf{A}_{12} \mathbf{A}_{22}^{-1} \mathbf{y}_t^2.$$