

Midterm Examination
 Answer Key

1. (Borrowing from an answer key by David Cook.) The four main equations of our ad hoc model are:

$$y_t = m_t - p_t - \theta_t, \quad (\text{AD})$$

$$y_t = p_t - w_t + \eta_t, \quad (\text{AS})$$

$$w_t = E(p_t | I_{t-1}^P), \quad (\text{WS})$$

$$m_t = \alpha \eta_t. \quad (\text{MS})$$

The random shocks $\theta_t \in \Theta$ and $\eta_t \in \mathbf{N}$ are both zero-mean and i.i.d., and are independent of each other at all leads and lags. Initially we assume that neither shock is present in the private agent's information set (I_{t-1}^P), but that the supply shock η_t is observed by the government:

$$I_{t-1}^g = \{I_{t-1}^P, \eta_t\}.$$

- (a) Combining (AD), (AS) and (WS) shows that

$$p_t = \frac{1}{2} [m_t + E(p_t | I_{t-1}^P) - (\theta_t + \eta_t)], \quad (\text{EQP})$$

$$y_t = \frac{1}{2} [m_t - E(p_t | I_{t-1}^P) + \eta_t - \theta_t]. \quad (\text{EQY})$$

Taking expectations on both sides of (EQP), and applying the law of iterated expectations shows that

$$E(p_t | I_{t-1}^P) = E(m_t - (\theta_t + \eta_t) | I_{t-1}^P). \quad (\text{CEP})$$

Recalling that θ_t and η_t are both zero-mean, i.i.d. and unobserved by private agents, and imposing (MS), (CEP) simplifies to

$$E(p_t | I_{t-1}^P) = 0. \quad (\text{CEP}')$$

- (b) Inserting (MS) and (CEP') into (EQY) yields

$$y_t = \frac{1}{2} [(\alpha + 1) \eta_t - \theta_t],$$

which implies that

$$V(y_t) = \frac{(\alpha + 1)^2}{4} \sigma_\eta^2 + \frac{1}{4} \sigma_\theta^2.$$

To minimize the variance of output, the monetary authority sets $\alpha = -1$, so that

$$y_t^U = -\frac{1}{2} \theta_t.$$

- (c) We now suppose that monetary authority announces its money supply decision before wages are set, so that the private agent's information set includes m_t . Given equation (MS), knowledge of m_t also implies knowledge of the supply shock η_t . It follows that we can rewrite equation (CEP) as

$$E(p_t | I_{t-1}^P) = m_t - \eta_t. \quad (\text{CEP}'')$$

Inserting (CEP'') into (EQY) yields

$$y_t^A = \frac{1}{2} [m_t - m_t + \eta_t + \eta_t - \theta_t] = \eta_t - \frac{1}{2}\theta_t.$$

- (d) Next we suppose that the monetary authority observes the demand shock θ_t after it sets m_t but prior to making any announcements. Suppose further that the monetary authority selectively announces m_t as way to further reduce the variance of output. We will assume for the moment that

$$y_t = \begin{cases} y_t^A, & \text{when } m_t \text{ is announced} \\ y_t^U, & \text{when } m_t \text{ is not announced} \end{cases}. \quad (\text{ANN})$$

Given that θ_t and η_t are both zero-mean variables, minimizing the variance of output boils down to minimizing squared output. Since the monetary authority announces at its discretion, it is choosing the announcement rule $A : \{\Theta \times \mathbf{N}\} \rightarrow \{0, 1\}$ that solves

$$\min_{A(\cdot)} E \left([A(\theta, \eta) y_t^A + [1 - A(\theta, \eta)] y_t^U]^2 \right) = \min_{A(\cdot)} E \left(A(\theta, \eta) [y_t^A]^2 + [1 - A(\theta, \eta)] [y_t^U]^2 \right),$$

where $A(\theta, \eta) = 1$ implies an announcement. The monetary authority will announce if

$$(y_t^A)^2 < (y_t^U)^2 \Leftrightarrow \left(\eta_t - \frac{1}{2}\theta_t \right)^2 < \left(-\frac{1}{2}\theta_t \right)^2 \Leftrightarrow |2\eta_t - \theta_t| < |\theta_t|,$$

which is to say it will announce the money supply if θ_t is the same sign as η_t and $|\eta_t| < |\theta_t|$.

- (e) The combination of equation (ANN) and the answer to part (d) is **not** consistent with equation (WS). In particular, our answer to part (d) showed that the monetary authority would announce its money supply only when θ_t and η_t are the same sign and $|\eta_t| < |\theta_t|$. Under rational expectations, private agents would use this information when predicting prices. However, y_t^A and y_t^U are based on the assumption that θ_t is completely unobserved, and thus ignore this additional information.

Note that proceeding as we did in part (d) violates the Lucas critique. In particular, y_t^U is based on the assumption that m_t is never announced, while y_t^A is based on the assumption that m_t is always announced. In both cases, private agents make forecasts consistent with the monetary authority's policy rule. However, the strategy described in part (d) is yet another policy rule, which would imply yet another forecast. The monetary authority, in choosing between y_t^A and y_t^U , is ignoring the way in which the adoption of a new policy rule would change private agents' forecasts, and thus the process for output itself.

2. We are considering a one-tree, one-good Lucas tree model with CRRA preferences, where the uncertainty is due to consumption taxes. Recall that p_t is the price of stock, and R_t^{-1} is the price of a one-period risk-free discount bond.

(a) Since τ_t summarizes the state of the aggregate economy, the consumer's problem can be written as

$$\begin{aligned} V(x_t, \tau_t) &= \max_{c_t \geq 0, s_{t+1}, b_{t+1}} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \beta E_t(V(x_{t+1}, \tau_{t+1})) \\ \text{s.t. } x_t &= (1 + \tau_t)c_t + p_t s_{t+1} + R_t^{-1} b_{t+1}, \\ x_{t+1} &= x(\tau_{t+1}) = [p(\tau_{t+1}) + d] s_{t+1} + b_{t+1} + T(\tau_{t+1}). \end{aligned}$$

Rewriting the consumer's problem as a Lagrangean, we get

$$\begin{aligned} V(x_t, \tau_t) &= \min_{\{\lambda_t \geq 0\}} \max_{\{c_t \geq 0, s_{t+1}, b_{t+1}\}} \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \lambda_t (x_t - (1 + \tau_t)c_t - p_t s_{t+1} - R_t^{-1} b_{t+1}) \\ &\quad + \beta E_t(V([p_{t+1} + d] s_{t+1} + b_{t+1} + T_{t+1}, \tau_{t+1})). \end{aligned}$$

The FOC for an interior solution are:

$$\begin{aligned} c_t^{-\sigma} &= \lambda_t (1 + \tau_t), \\ \lambda_t p_t &= \beta E_t \left(\frac{\partial V[t+1]}{\partial x_{t+1}} [p_{t+1} + d] \right), \\ \lambda_t R_t^{-1} &= \beta E_t \left(\frac{\partial V[t+1]}{\partial x_{t+1}} \right). \end{aligned}$$

Since (following Benveniste-Scheinkman),

$$\frac{\partial V[t]}{\partial x_t} = \lambda_t = c_t^{-\sigma} \frac{1}{1 + \tau_t},$$

the Euler equations are

$$R_t^{-1} c_t^{-\sigma} \frac{1}{1 + \tau_t} = \beta E_t \left(c_{t+1}^{-\sigma} \frac{1}{1 + \tau_{t+1}} \right), \quad (\text{EE1})$$

$$p_t c_t^{-\sigma} \frac{1}{1 + \tau_t} = \beta E_t \left(c_{t+1}^{-\sigma} [p_{t+1} + d] \frac{1}{1 + \tau_{t+1}} \right). \quad (\text{EE2})$$

(b) Given the random variable τ_0 , the conditional density $f(\tau_{t+1} | \tau_t)$, and the initial endowments $s_0 = 1$ and $b_0 = 0$, a recursive rational expectations equilibrium consists of pricing functions $p(\tau)$ and $R^{-1}(\tau)$, a transfer function $T(\tau)$, a value function $V(x, \tau)$, and decision functions $c(x, \tau)$, $s(x, \tau)$, and $b(x, \tau)$ such that:

1. Given the pricing functions $p(\tau)$ and $R^{-1}(\tau)$ and the transfer function $T(\tau)$, the value and policy functions $V(x, \tau)$, $c(x, \tau)$, $s(x, \tau)$, and $b(x, \tau)$ solve the consumer's problem.
2. The government's budget constraint is satisfied: $T(\tau) = \tau \cdot c(x, \tau)$.
3. Markets clear: for $x = p(\tau) + d + \tau d$, $c(x, \tau) = d$, $s(x, \tau) = 1$, and $b(x, \tau) = 0$.

- (c) To achieve equilibrium, we impose $c_t = d_t$, $s_{t+1} = 1$, $T_t = \tau_t d$ and $b_{t+1} = 0$. Then equilibrium bond prices and returns follow

$$\begin{aligned} R_t^{-1} &= \beta E_t \left(\frac{1 + \tau_t}{1 + \tau_{t+1}} \right) & (\text{EE1}') \\ \Rightarrow R_t &= \frac{[\beta(1 + \tau_t)]^{-1}}{E_t([1 + \tau_{t+1}]^{-1})}. \end{aligned}$$

- (d) If the consumer faces a constant tax rate, the risk-free interest rate is

$$R_t = \beta^{-1} \frac{[1 + \tau]^{-1}}{E_t([1 + \tau]^{-1})} = \beta^{-1},$$

and the tax has no effect. This is because the consumption tax affects a consumer's willingness to save only by modifying the cost of shifting goods across time. Holding interest rates fixed, if next period's consumption tax is higher than this period's tax ($\tau_{t+1} > \tau_t$), a unit of income next period will purchase fewer consumption goods than a unit of income this period. This reduces the consumer's willingness to save. In an endowment economy, interest rates will increase to keep equilibrium savings at 0 ($c_t = d$). However, when consumption taxes are the same in both periods, the rate at which income is converted to consumption is constant, and consumption taxes have no effect on interest rates.

- (e) The expected rate of return on stocks is

$$E_t(R_t^S) = E_t \left(\frac{p_{t+1} + d_{t+1}}{p_t} \right).$$

Rewriting equation (EE2), and imposing equilibrium, yields:

$$\begin{aligned} 1 &= \beta E_t \left(\frac{1 + \tau_t}{1 + \tau_{t+1}} \left(\frac{p_{t+1} + d}{p_t} \right) \right) \\ &= \beta E_t \left(\frac{1 + \tau_t}{1 + \tau_{t+1}} R_t^S \right) \\ &= \beta E_t \left(\frac{1 + \tau_t}{1 + \tau_{t+1}} \right) E_t(R_t^S) + Cov_t \left(\beta \frac{1 + \tau_t}{1 + \tau_{t+1}}, R_t^S \right). \end{aligned}$$

Inserting equation (EE1') and rearranging, we get:

$$\begin{aligned} 1 &= R_t^{-1} E_t(R_t^S) + Cov_t \left(\beta \frac{1 + \tau_t}{1 + \tau_{t+1}}, R_t^S \right), \\ E_t(R_t^S) &= R_t - R_t Cov_t \left(\beta \frac{1 + \tau_t}{1 + \tau_{t+1}}, R_t^S \right), \\ &= R_t - \frac{[1 + \tau_t]^{-1}}{\beta E_t([1 + \tau_{t+1}]^{-1})} Cov_t \left(\beta \frac{1 + \tau_t}{1 + \tau_{t+1}}, R_t^S \right), \\ &= R_t + \frac{Cov_t(-[1 + \tau_{t+1}]^{-1}, R_t^S)}{E_t([1 + \tau_{t+1}]^{-1})} \\ &\equiv R_t + e_t, \end{aligned}$$

so that the expected return on stocks equals the expected return on the risk-free bond plus the risk-premium e_t . To understand e_t note that, holding the marginal utility of consumption constant, a consumer would most like an asset that yielded a high return when consumption taxes were low and consumption was cheapest. But having a lower value of $1 + \tau_{t+1}$ is equivalent to having a lower (more negative) value of $-[1 + \tau_{t+1}]^{-1}$. Hence if the correlation between consumption taxes and returns is positive, stockholders will require a higher rate of return.

(In fact, it can be shown that $p_t = d(1 + \tau_t) E_t \left(\sum_{j=1}^{\infty} (1 + \tau_{t+j})^{-1} \right)$, which for i.i.d. taxes would imply that $e_t > 0$.)