Economics 701: Macroeconomics II
Spring 2007
Professor Jones

Final Examination
Answer Key

We are analyzing an stochastic growth model with home production.

Recall that the representative producer solves

$$\max_{L_{Mt}\geq 0} \quad \Pi_t = Y_t - W_t L_{Mt},$$

$$Y_t = \frac{1}{1-\alpha} L_{Mt}^{1-\alpha}, \quad 0 < \alpha < 1, \qquad \text{(PRF)}$$

while the representative consumer solves

$$\max_{\{C_{Mt},C_{Ht},K_{t+1},L_{Mt},L_{tH},\}_{t=0}^{\infty}} E_0\left(\sum\nolimits_{t=0}^{\infty} \beta^t \left[\ln(C_{Mt}) + Z_t \ln(C_{Ht}) - \frac{\chi}{1+\gamma}(L_{Mt}+L_{Ht})^{1+\gamma}\right]\right),$$

$$0 < \beta < 1, \quad \gamma \geq 0, \quad \chi > 0,$$

$$s.t. \quad C_{Mt} + K_{t+1} = (1+r) K_t + W_t L_{Mt} + \Pi_t, \qquad \text{(FBC)}$$

$$C_{Ht} = \frac{1}{1-\eta} L_{Ht}^{1-\eta}, \qquad \text{(HP)}$$

along with the other usual boundary and non-negativity constraints. Finally, the log of the taste shock Z_t follows

$$\widehat{z}_t \equiv \ln(Z_t/Z_{SS}) = \phi \widehat{z}_{t-1} + \varepsilon_t, \quad Z_{ss} > 0, \quad 0 \leq \phi < 1, \qquad \text{(TS)}$$

where $\{\varepsilon_t\}$ is an exogenous stationary martingale difference sequence.

1. (12 points) The consumer's problem can be written as a Lagrangean

$$\mathcal{L} = E_0\Bigg(\sum\nolimits_{t=0}^{\infty} \beta^t \Bigg[\ln(C_{Mt}) + Z_t \ln(C_{Ht}) - \frac{\chi}{1+\gamma}(L_{Mt}+L_{Ht})^{1+\gamma}$$
$$+ \lambda_{Mt}\left[(1+r)K_t + W_t L_{Mt} + \Pi_t - C_{Mt} - K_{t+1}\right].$$
$$+ \lambda_{Ht}\left[\frac{1}{1-\eta} L_{Ht}^{1-\eta} - C_{Ht}\right]\Bigg]\Bigg)$$

The first-order conditions for an interior solution are

$$\begin{aligned}
\frac{1}{C_{Mt}} &= \lambda_{Mt}, \\
Z_t \frac{1}{C_{Ht}} &= \lambda_{Ht} \\
\lambda_{Mt} W_t &= \chi (L_{Mt}+L_{Ht})^{\gamma}, \\
\lambda_{Ht} L_{Ht}^{-\eta} &= \chi (L_{Mt}+L_{Ht})^{\gamma}, \\
\lambda_{Mt} &= \beta E_t\left((1+r)\lambda_{M,t+1}\right).
\end{aligned}$$

Eliminating the multipliers, and imposing $\beta(1+r)=1$, yields

$$\begin{aligned} W_t \frac{1}{C_{Mt}} &= \chi (L_{Mt}+L_{Ht})^{\gamma}, && \text{(LL1)} \\ Z_t L_{Ht}^{-\eta} \frac{1}{C_{Ht}} &= \chi (L_{Mt}+L_{Ht})^{\gamma}, && \text{(LL2)} \\ \frac{1}{C_{Mt}} &= E_t\left(\frac{1}{C_{M,t+1}}\right). && \text{(EE)} \end{aligned}$$

2. (8 points) The first order condition for profit maximization is

$$L_{Mt}^{-\alpha} = W_t. \tag{PM}$$

Inserting this result into equation (LL1), we get

$$L_{Mt}^{-\alpha}\frac{1}{C_{Mt}} = \chi (L_{Mt}+L_{Ht})^{\gamma}.$$

Inserting (LL2) and (HP) produces

$$\begin{aligned} L_{Mt}^{-\alpha}\frac{1}{C_{Mt}} &= Z_t L_{Ht}^{-\eta}\frac{1}{C_{Ht}} \\ &= Z_t L_{Ht}^{-\eta}\frac{1}{\frac{1}{1-\eta}L_{Ht}^{1-\eta}} = Z_t(1-\eta)L_{Ht}^{-1}, \end{aligned}$$

which implies that

$$L_{Ht} = Z_t(1-\eta)C_{Mt}L_{Mt}^{\alpha}. \tag{HM}$$

3. (8 points) Using equation (HM), we can rewrite equation (LL1) as

$$L_{Mt}^{-\alpha}\frac{1}{C_{Mt}} = \chi\left(L_{Mt}+Z_t(1-\eta)C_{Mt}L_{Mt}^{\alpha}\right)^{\gamma}. \tag{LL1$'$}$$

The Euler equation was derived in part (1). The capital accumulation equation can be written as

$$K_{t+1} = (1+r)K_t + Y_t - C_t, \tag{CA}$$

with Y_t given by equation (PRF). This constraint can either be derived directly as a resource constraint, or found by combining the consumer's budget constraint with the definition of profits.

4. (38 points) We first log-linearize the labor market equation.

   (a) Logging both sides of equation (LL1$'$) yields

$$\begin{aligned} &-\alpha\ln L_{Mt} - \ln C_{Mt} = \\ &\qquad \ln(\chi) + \gamma\ln(L_{Mt}+L_{Ht}) \\ &\qquad \ln(\chi) + \gamma\ln\left[\exp(\ln L_{Mt}) + \exp(\ln L_{Ht})\right] \\ &\qquad \ln(\chi) + \gamma\ln\left[\exp(\ln L_{Mt}) + \exp(\ln Z_t + \ln(1-\eta) + \ln C_{Mt} + \alpha\ln L_{Mt})\right]. \end{aligned}$$

Implicitly differentiating this expression yields

$$-\alpha d\ln L_{Mt} - d\ln C_{Mt} = \gamma \frac{1}{L_{Mt}+L_{Ht}} \left[L_{Mt} d\ln L_{Mt} + L_{Ht} \left(d\ln Z_t + d\ln C_{Mt} + \alpha d L_{Mt} \right) \right].$$

Let lower-case letters with carats "$\widehat{\ }$" denote deviations of logged variables around their steady state values. The preceding equation simplifies to

$$-\alpha \widehat{\ell}_{Mt} - \widehat{c}_{Mt} \approx \frac{\gamma}{L_{SS}} \left[L_{MSS} \widehat{\ell}_{Mt} + L_{HSS} \left(\widehat{z}_t + \widehat{c}_{Mt} + \alpha \widehat{\ell}_{Mt} \right) \right].$$

Collect terms and rearrange

$$-\left[\alpha + \frac{\gamma}{L_{SS}} L_{MSS} + \frac{\gamma\alpha}{L_{SS}} L_{HSS} \right] \widehat{\ell}_{Mt} = \left[1 + \frac{\gamma}{L_{SS}} L_{HSS} \right] \widehat{c}_{Mt} + \frac{\gamma}{L_{SS}} L_{HSS} \widehat{z}_t,$$

so that

$$\begin{aligned} \widehat{\ell}_{Mt} &= -\left[\lambda_1 \widehat{z}_t + \theta_1 \widehat{c}_{Mt} \right], \qquad \text{(LL1}''\text{)} \\ \lambda_1 &\equiv \frac{\gamma}{L_{SS}} L_{HSS} \Big/ \left[\alpha + \frac{\gamma}{L_{SS}} L_{MSS} + \frac{\gamma\alpha}{L_{SS}} L_{HSS} \right] > 0, \\ \theta_1 &\equiv \left[1 + \frac{\gamma}{L_{SS}} L_{HSS} \right] \Big/ \left[\alpha + \frac{\gamma}{L_{SS}} L_{MSS} + \frac{\gamma\alpha}{L_{SS}} L_{HSS} \right] \\ &= \lambda_1 + 1 \Big/ \left[\alpha + \frac{\gamma}{L_{SS}} L_{MSS} + \frac{\gamma\alpha}{L_{SS}} L_{HSS} \right] > \lambda_1 > 0. \end{aligned}$$

Proceeding similarly, it follows from equation (PRF′) that

$$\exp(\ln Y_t) = \frac{1}{1-\alpha} \exp\left((1-\alpha) \ln L_{Mt} \right),$$

so that

$$\widehat{y}_t = (1-\alpha) \widehat{\ell}_{Mt} = -(1-\alpha) \left[\lambda_1 \widehat{z}_t + \theta_1 \widehat{c}_{Mt} \right] \equiv -\left[\lambda_2 \widehat{z}_t + \theta_2 \widehat{c}_{Mt} \right]. \qquad \text{(PRF}''\text{)}$$

With $\alpha \in (0,1)$, λ_2 and θ_2 are both positive.

(b) If λ_1 and θ_1 are both positive, it follows from equation (LL″) that market hours are decreasing both in the preference shock, $\widehat{z}_t$, and in market consumption, $\widehat{c}_{Mt}$. When $\widehat{z}_t$ increases, the agent has a stronger taste for home-produced goods, leading him to devote more time to home production. The additional time devoted to home production reduces available leisure and increases the marginal disutility of work, causing market hours to decrease. The interaction of hours through the time constraint is essential: if preferences were separable in home and market hours as well as in home and market consumption, home production shocks would have no effect on market hours. An increase in $\widehat{c}_{Mt}$ reduces market hours through a standard wealth effect. As $\widehat{c}_{Mt}$ increases, the marginal utility of consumption and hence the marginal benefit to work decreases.

(c) Letting $f_{SS} \equiv L_{HSS}/L_{SS}$ denote the steady-state ratio of home-production labor to total labor, we can rewrite λ_1 as

$$\lambda_1 \equiv f_{SS} \frac{1}{\alpha + \gamma (1 - f_{SS}) + \gamma \alpha f_{SS}} = \frac{f_{SS}}{\alpha + \gamma - \gamma (1 - \alpha) f_{SS}},$$

which is increasing in f_{SS}. As f_{SS} increases, home production hours become larger and larger relative to market production hours, so that a small movement of hours into or out of home production causes a large proportional change in market hours. Since or log-linearization delivers elasticities, a small change in the absolute number of hours that is a big proportional change will generate a large coefficient.

(d) Our answer to part (c) suggested that market hours will be more responsive in countries that have a high ratio of home to market production. Since developing countries typically have more informal/home production, especially by farmers, market hours in these countries should be more sensitive to home production shocks.

5. (18 points) Suppose that the steady state consumption-to-capital ratio, C_{MSS}/K, is ψ, with $\psi > r$. Using this fact, it is straightforward, if tedious, to show that the log-linearized capital accumulation equation is

$$\begin{aligned} \widehat{k}_{t+1} &= (1 + r) \widehat{k}_t - \omega_1 \widehat{z}_t - \omega_2 \widehat{c}_{Mt}, \\ \omega_1 &= \lambda_2 (\psi - r) > 0, \quad \omega_2 = \theta_2 (\psi - r) + \psi > \omega_1 + \psi > \omega_1. \end{aligned} \tag{CA$'$}$$

It is also straightforward to show that the log-linearized Euler equation is:

$$E_t (\widehat{c}_{M,t+1}) = \widehat{c}_{Mt}. \tag{EE$'$}$$

It can be shown that the system given by equations (CA′), (EE′) and (TS) is saddle-path stable, so that consumption can be written as a function of capital and technology:

$$\widehat{c}_{Mt} = \pi \widehat{k}_t - \mu \widehat{z}_t, \tag{CF}$$

(a) Combining equations (CF) and equation (CA′) yields

$$\widehat{k}_{t+1} = (1 + r - \omega_2 \pi) \widehat{k}_t - (\omega_1 - \omega_2 \mu) \widehat{z}_t. \tag{CA$''$}$$

(b) Inserting equation (CF) into the *right*-hand-side of equation (EE′) yields

$$E_t (\widehat{c}_{t+1}) = \pi \widehat{k}_t - \mu \widehat{z}_t.$$

(c) Inserting equation (CF) into the *left*-hand-side of equation (EE′) yields

$$E_t (\widehat{c}_{t+1}) = E_t \left(\pi \widehat{k}_{t+1} - \mu \widehat{z}_{t+1} \right).$$

Inserting (CA″) and (TS) into this expression yields

$$\begin{aligned} E_t (\widehat{c}_{t+1}) &= \pi \left[(1 + r - \omega_2 \pi) \widehat{k}_t - (\omega_1 - \omega_2 \mu) \widehat{z}_t \right] - \mu \phi \widehat{z}_t, \\ &= \pi (1 + r - \omega_2 \pi) \widehat{k}_t - [\pi (\omega_1 - \omega_2 \mu) + \mu \phi] \widehat{z}_t. \end{aligned}$$

(d) If the expressions for $E_t(\widehat{c}_{t+1})$ in parts (b) and (c) are to be equal,

$$\begin{aligned} \pi(1+r-\omega_2\pi) &= \pi, \\ \pi(\omega_1-\omega_2\mu)+\mu\phi &= \mu. \end{aligned}$$

The first of these two equations has the non-zero solution

$$1+r-\omega_2\pi = 1 \Rightarrow \pi = \frac{r}{\omega_2},$$

while the second equation implies that

$$\pi\omega_1 - r\mu + \mu\phi = \mu,$$

so that

$$\begin{aligned} \mu &= \pi\frac{\omega_1}{(1-\phi+r)} \\ &= \frac{r\omega_1}{\omega_2(1-\phi+r)}. \end{aligned}$$

6. (16 points) Now we consider the average product of labor.

(a) It immediately follows from equation (LL1″) that market labor follows

$$\begin{aligned} \widehat{\ell}_{Mt} &= -\left[\lambda_1\widehat{z}_t + \theta_1\left(\pi\widehat{k}_t - \mu\widehat{z}_t\right)\right] \\ &= -\left[(\lambda_1-\mu\theta_1)\widehat{z}_t + \theta_1\pi\widehat{k}_t\right], \end{aligned}$$

while equation (PRF″) implies that market output follows

$$\widehat{y}_t = (1-\alpha)\widehat{\ell}_{Mt} = -(1-\alpha)\left[(\lambda_1-\mu\theta_1)\widehat{z}_t + \theta_1\pi\widehat{k}_t\right]$$

Then the log of average market labor productivity is

$$\widehat{apl}_t = \widehat{y}_t - \widehat{\ell}_{Mt} = -\alpha\widehat{\ell}_{Mt} = \alpha\left[(\lambda_1-\mu\theta_1)\widehat{z}_t + \theta_1\pi\widehat{k}_t\right].$$

If labor and consumption are normal goods, the substitution effects of the home production shock dominate the income effects, so that $\lambda_1 - \mu\theta_1 > 0$. As background (not required in the exam), one can show this more rigorously:

$$\begin{aligned} \mu\theta_1 &= \frac{r\omega_1}{\omega_2(1-\phi+r)}\theta_1 = \left(\frac{r}{1-\phi+r}\right)\left(\frac{\omega_1}{\omega_2}\right)\theta_1 \\ &= \left(\frac{r}{1-\phi+r}\right)\left(\frac{\lambda_2(\psi-r)}{\theta_2(\psi-r)+\psi}\right)\theta_1 \\ &= \left(\frac{r}{1-\phi+r}\right)\left(\frac{\psi-r}{\theta_2[\psi-r+\psi/\theta_2]}\right)\lambda_1\theta_2 \\ &= \left(\frac{r}{1-\phi+r}\right)\left(\frac{\psi-r}{[\psi-r+\psi/\theta_2]}\right)\lambda_1 < \lambda_1, \end{aligned}$$

with the last line following from $\phi < 1$, $\psi > r > 0$, and $\theta_2 > 0$.

(b) We see from part (a) that when $\widehat{z}$ rises, market output and market labor both initially fall, while average productivity initially rises. Extrapolating from the initial response, one would expect average labor productivity to move in the direction opposite of market output, so that it is counter-cyclical. (Because home production is rarely observed, "pro-" and "counter-"cyclical are defined in terms of market output.) Intuitively, an increase in the demand for home-produced goods leisure shifts the market labor supply curve in, causing Y and L_M to both decrease. But with decreasing returns to market labor, a decrease in labor causes average productivity, Y/L_M, to increase.