Economics 701: Macroeconomics II
Spring 2006
Professor Jones

Midterm Examination
Answer Key

1. The main equations of our ad hoc model are:

$$y_t = m_t + \theta_t - p_t, \tag{AD}$$
$$y_t = p_t - \frac{1}{2}w_t - \frac{1}{2}w_{t-1}, \tag{AS}$$
$$w_t = E\left(p_t | I_{t-1}\right), \tag{WS}$$
$$\theta_t = \varepsilon_t + \phi\varepsilon_{t-1}, \tag{TS}$$
$$m_t = \beta\varepsilon_{t-1}. \tag{MS}$$

(a) Combining equations (AD), (AS) and (WS) shows that

$$p_t = \frac{1}{2}\left[m_t + \theta_t + \frac{1}{2}w_t + \frac{1}{2}w_{t-1}\right], \tag{EQP}$$
$$y_t = \frac{1}{2}\left[m_t + \theta_t - \frac{1}{2}w_t - \frac{1}{2}w_{t-1}\right]. \tag{EQY}$$

(b) Let's solve for w_t:

1. Taking expectations on both sides of equation (EQP), and applying the law of iterated expectations yields

$$\begin{aligned} E\left(p_t | I_{t-1}\right) &= \frac{1}{2}E\left(m_t + \theta_t + \frac{1}{2}E\left(p_t | I_{t-1}\right) + \frac{1}{2}E\left(p_{t-1} | I_{t-2}\right) \middle| I_{t-1}\right) \\ &= \frac{4}{3}\cdot\frac{1}{2}E\left(m_t + \theta_t + \frac{1}{2}E\left(p_{t-1} | I_{t-2}\right) \middle| I_{t-1}\right) \\ &= \frac{2}{3}E\left(\beta\varepsilon_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1} + \frac{1}{2}E\left(p_{t-1} | I_{t-2}\right) \middle| I_{t-1}\right). \end{aligned}$$

Note that $E\left(p_{t-2} | I_{t-2}\right)$ is **known** at time $t-1$. The last line follows from equations (MS) and (TS). Recalling that ε_t is i.i.d. produces

$$E\left(p_t | I_{t-1}\right) = \frac{2}{3}\left(\beta + \phi\right)\varepsilon_{t-1} + \frac{1}{3}E\left(p_{t-1} | I_{t-2}\right).$$

2. Recalling $w_t = E\left(p_t | I_{t-1}\right)$, we have

$$w_t = \frac{1}{3}w_{t-1} + \frac{2}{3}\left(\beta + \phi\right)\varepsilon_{t-1} \tag{WS$'$}$$
$$\Leftrightarrow \left(1 - \frac{1}{3}L\right)w_t = \frac{2}{3}\left(\beta + \phi\right)\varepsilon_{t-1}.$$

3. Assuming that $\lim_{J\to\infty} \left(\frac{1}{3}\right)^J w_{t-J} = 0$—which will turn out to be true—we can invert the lag operator to get:

$$w_t = \frac{2}{3}(\beta + \phi) \sum_{j=0}^{\infty} \left(\frac{1}{3}\right)^j \varepsilon_{t-1-j}.$$

(c) (As several students noted) it follows from equation (WS′) that

$$3w_t = w_{t-1} + 2(\beta + \phi)\varepsilon_{t-1},$$

which in turn implies that

$$w_t + w_{t-1} = 4w_t - 2(\beta + \phi)\varepsilon_{t-1}.$$

Inserting this result into equation (EQP), we see that prices follow

$$\begin{aligned} p_t &= \frac{1}{2}\left[\beta\varepsilon_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1} + \frac{1}{2}(4w_t - 2(\beta+\phi)\varepsilon_{t-1})\right] \\ &= \frac{1}{2}[\varepsilon_t + 2w_t] \\ &= \frac{1}{2}\varepsilon_t + w_t, \end{aligned} \qquad \text{(EQP′)}$$

while output follows

$$\begin{aligned} y_t &= \frac{1}{2}\left[\beta\varepsilon_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1} - \frac{1}{2}(4w_t - 2(\beta+\phi)\varepsilon_{t-1})\right] \\ &= \frac{1}{2}[\varepsilon_t - 2w_t + 2(\beta+\phi)\varepsilon_{t-1}] \\ &= \frac{1}{2}\varepsilon_t + (\beta+\phi)\varepsilon_{t-1} - w_t. \end{aligned} \qquad \text{(EQY′)}$$

(d) Becasue w_t is an infinite sum of lagged values of ε_t, output and prices depend on all these lagged values as well, even though θ_t depends only on ε_t and ε_{t-1}. This reflects the staggered wage contracts. ε_{t-1} first affects wages in time t, by entering into w_t. However, ε_{t-1} and w_t also affects output and prices in time $t+1$, as half of the economy uses a lagged wage. This in turn implies that ε_{t-1} and w_t will enter into forecasts of p_{t+1}, and thus affect w_{t+1}, which will affect output and prices at time $t+2$, and so on.

(e) It is straightforward to see that setting $\beta = -\phi$ will set $w_t = 0$, and $y_t = \frac{1}{2}\varepsilon_t$. Since the monetary authority by assumption cannot react to ε_t at time t, this clearly minimizes output variance.

2. We are considering a standard Lucas tree model with discrete shocks. The preferences of the representative consumer are

$$E_0\left(\sum_{t=0}^{\infty}\beta^t\frac{1}{1-\sigma}\left[c_t^{1-\sigma}-1\right]\right),\quad 0<\beta<1,\quad \sigma>0.$$

The economy starts off with each consumer owning one dividend-producing tree. Dividends can take on two values, d_L and $d_H > d_L$, and follow a symmetric two-state Markov process:

$$\begin{aligned}f(d_L,d_L) &= \Pr\left(d_{t+1}=d_L \mid d_t=d_L\right)=\pi=f(d_H,d_H),\\ f(d_H,d_L) &= \Pr\left(d_{t+1}=d_H \mid d_t=d_L\right)=1-\pi=f(d_L,d_H).\end{aligned}$$

(a) Let $q(d',d)$ denote the price of the one-step-ahead contingent claim that pays off when $d_{t+1}=d'$. Writing the consumer's problem as a Lagrangean, we get

$$\begin{aligned}&V(x_t,d_t)=\\ &\min_{\lambda_t\geq 0}\max_{c_t\geq 0,\ s_{t+1},\ z(d')}\frac{c_t^{1-\sigma}-1}{1-\sigma}+\lambda_t\left(x_t-c_t-p_ts_{t+1}-\sum_{d'\in\{d_L,d_H\}}q(d',d_t)\,z(d')\right)\\ &\quad+\beta\sum_{d_{t+1}\in\{d_L,d_H\}}f(d_{t+1},d_t)\times V\big(z(d_{t+1})+\left[p_{t+1}(d_{t+1})+d_{t+1}\right]s_{t+1},d_{t+1}\big),\end{aligned}$$

The FOC for an interior solution are:

$$\begin{aligned}c_t^{-\sigma}&=\lambda_t,\\ \lambda_tp_t&=\beta\sum_{d_{t+1}\in\{d_L,d_H\}}f(d_{t+1},d_t)\times\frac{\partial V\left(x_{t+1}(d'),d'\right)}{\partial x_{t+1}}\left[p_{t+1}(d_{t+1})+d_{t+1}\right],\\ \lambda_tq(d',d_t)&=\beta\frac{\partial V\left(x_{t+1}(d'),d'\right)}{\partial x_{t+1}}f(d',d_t),\quad d'\in\{d_L,d_H\}.\end{aligned}$$

Since (following Benveniste-Scheinkman),

$$\frac{\partial V[t]}{\partial x_t}=\lambda_t,$$

the Euler equations are

$$p_tc_t^{-\sigma}=\beta E_t\left(c_{t+1}(d_{t+1})^{-\sigma}\left[p_{t+1}(d_{t+1})+d_{t+1}\right]\right), \tag{EE1}$$

$$q(d',d_t)=\beta\left(\frac{c_t}{c_{t+1}(d')}\right)^{\sigma}f(d',d_t),\qquad d'\in d'\in\{d_L,d_H\}. \tag{EE2}$$

(b) To achieve equilibrium,we impose $s_{t+1}=1$, $z(d_{t+1})=0$, $\forall d_{t+1}$ so that $c_t=d_t$. It follows from equation (EE2) that the equilibrium price for a contingent claim is

$$q(d',d_t)=\beta\left(\frac{d_t}{d'}\right)^{\sigma}f(d',d_t).$$

Continuing, we get

$$q(d_H, d_L) = \beta \left(\frac{d_L}{d_H} \right)^{\sigma} (1 - \pi),$$
$$q(d_L, d_H) = \beta \left(\frac{d_H}{d_L} \right)^{\sigma} (1 - \pi).$$

With $d_H > d_L$, it follows from $\sigma > 0$ that $q(d_L, d_H)$ is larger. In this configuration, the marginal utility of current consumption is low—as output/consumption $(= d_H)$ is high—while the marginal utility of future consumption is high. Consumption smoothing agents are thus very eager to to trade current consumption for consumption in this future state, leading to a high price. The other contingent claim, which requires agents to trade current consumption with a high marginal value for future consumption with a low marginal value, has a lower price.

(c) In equilibrium, $c_t = d_t$. Inserting this result into equation (EE1) shows that in equilibrium the price of a share of stock obeys

$$E_t \left\{ \left(1 - \beta L^{-1}\right) d_t^{-\sigma} p_t \right\} = E_t \left(d_{t+1}^{1-\sigma} \right),$$

where L denotes the lag operator. Inverting $(1 - \beta L^{-1})$, and imposing

$$\lim_{J \to \infty} \beta^J E_t \left(d_{t+J}^{-\sigma} p_{t+J} \right) = 0,$$

(the transversality condition), one gets

$$d_t^{-\sigma} p_t = E_t \left(\sum\nolimits_{j=1}^{\infty} \beta^j d_{t+j}^{1-\sigma} \right),$$
$$\Rightarrow p_t = d_t^{\sigma} E_t \left(\sum\nolimits_{j=1}^{\infty} \beta^j d_{t+j}^{1-\sigma} \right).$$

(d) It follows from arbitrage arguments that if an asset pays $w(d)$ units of consumption goods when $d_{t+1} = d$, its price is

$$p_t^w = \sum_{d_{t+1} \in \{d_L, d_H\}} w(d_{t+1})\, q(d_{t+1}, d_t).$$

When the asset is a risk-free discount bond, $w(d) = 1$. Imposing equation (EE2), it follows that the price of a risk-free bond, $R^{-1}(d_t)$ is given by

$$R^{-1}(d_t) = \beta \sum_{d_{t+1}} f(d_{t+1}, d_t) \left(\frac{d_t}{d_{+1}} \right)^{\sigma} = \beta d_t^{\sigma} E_t \left(d_{t+1}^{-\sigma} \right).$$

Continuing, we have

$$R^{-1}(d_L) = \beta \left[\pi \left(\frac{d_L}{d_L} \right)^{\sigma} + (1 - \pi) \left(\frac{d_L}{d_H} \right)^{\sigma} \right] < \beta \left[\pi + (1 - \pi) \right] = \beta.$$

This price is less than the discount rate β because of the possibility that future dividends take on the high value $d_H > d_L$. Marginal utility in this future state is lower than current marginal utility, which lowers the value of the bond.