Economics 701: Macroeconomics II
Spring 2005
Professor Jones

Midterm Examination
Answer Key

1. We are considering a variant of Lucas tree model , where there are two types of trees: apple trees, which will be indexed by "a"; and banana trees, which will be indexed by "b". Consumers are indifferent between consuming a unit of apple fruit or a unit of banana fruit; preferences are defined over total consumption, c_t. The dividend processes for the two trees follow

$$\begin{aligned} d_t^a &= \widehat{d}_t + \varepsilon_t^a, \\ d_t^b &= \widehat{d}_t + \varepsilon_t^b, \end{aligned}$$

where $\widehat{d}_t$ follows a time-invariant Markov process. The idiosyncratic shocks ε_t^a and ε_t^b share the same zero-mean i.i.d. distribution, but are mutually independent of each other and $\widehat{d}_t$ at all leads and lags. The economy starts off with half of the households owning one apple tree apiece, and the other half owning one banana tree. In addition, initial dividends are equivalent: $d_0^a = d_0^b$.

   (a) Let c^a (c^b) denote the consumption of the representative agent initially endowed with an apple (banana) tree. Given equal weights, the social planner's problem can be written as

$$\max_{\{c_t^a, c_t^b\}} E_0 \left(\sum_{t=0}^{\infty} \beta^t \left[\frac{1}{2} \left(f c_t^a - \frac{g}{2} (c_t^a)^2 \right) + \frac{1}{2} \left(f c_t^b - \frac{g}{2} \left(c_t^b \right)^2 \right) \right] \right),$$

$$s.t. \quad c_t^a + c_t^b \leq d_t^a + d_t^b.$$

   Assuming interiority, it immediately follows that the optimal allocation is

$$c_t^a = c_t^b = d_t \equiv \frac{1}{2} \left[d_t^a + d_t^b \right].$$

   This allocation can be achieved by letting each consumer hold 1/2 share of apple and 1/2 share of banana stock. Given that consumers have identical initial endowments and identical preferences, the social planner's allocation is also the competitive equilibrium.

   (b) We have just shown that in equilibrium, all consumers hold 1/2 share of each stock. This means that total consumption equals $d_t \equiv \frac{1}{2}[d_t^a + d_t^b]$. However, $\widehat{d}_t$ is the only part of d_t that has predictive content, as ε_t^a and ε_t^b are i.i.d. It follows that d_t and $\widehat{d}_t$ together summarize the state of the aggregate economy. In Lagrangean form, Bellman's functional equation is:

$$V(x_t, d_t, \widehat{d}_t) = \min_{\lambda_t \geq 0} \max_{c_t \geq 0,\ s_{t+1}^a,\ s_{t+1}^b} f c_t - \frac{g}{2} c_t^2 + \lambda_t \left(x_t - c_t - p_t^a s_{t+1}^a - p_t^b s_{t+1}^b \right) +$$
$$\beta E \left(V \left([p^a(d_{t+1}, \widehat{d}_{t+1}) + d_{t+1}^a] s_{t+1}^a + [p^b(d_{t+1}, \widehat{d}_{t+1}) + d_{t+1}^b] s_{t+1}^b, d_{t+1}, \widehat{d}_{t+1} \,\middle|\, \widehat{d}_t \right) \right).$$

Because the consumer's actions have no effect on the aggregate economy, his financial resources, x_t are a state variable for him but not for the aggregate economy. The first-order conditions for an interior solution are

$$\begin{aligned}
\lambda_t &= f - gc_t, \\
\lambda_t p_t^a &= \beta E_t \left(\frac{\partial V[t+1]}{\partial x_{t+1}} \left(p_{t+1}^a + d_{t+1}^a \right) \right), \\
\lambda_t p_t^b &= \beta E_t \left(\frac{\partial V[t+1]}{\partial x_{t+1}} \left(p_{t+1}^b + d_{t+1}^b \right) \right).
\end{aligned}$$

From Benveniste and Scheinkman's results, we have

$$\frac{\partial V[t]}{\partial x_t} = \lambda_t,$$

so that the first-order conditions reduce to:

$$p_t^i = \frac{1}{f - gc_t} \beta E_t \left((f - gc_{t+1}) \left(p_{t+1}^i + d_{t+1}^i \right) \right), \quad i \in \{a, b\}. \tag{EE}$$

(c) In equilibrium, $c_t = d_t$. Inserting this result into equation (EE) shows that in equilibrium the price of apple stock obeys

$$E_t \left\{ \left(1 - \beta L^{-1} \right) p_t^i (f - gd_t) \right\} = \beta E_t \left((f - gd_{t+1}) d_{t+1}^i \right),$$

where L denotes the lag operator. Inverting $(1 - \beta L^{-1})$, and imposing

$$\lim_{J \to \infty} \beta^J E_t \left(p_{t+J}^i (f - gd_{t+J}) \right) = 0,$$

(the transversality condition), one gets

$$p_t^i (f - gd_t) = E_t \left(\sum_{j=1}^{\infty} \beta^j (f - gd_{t+j}) d_{t+j}^i \right).$$

Recalling the expressions for d_{t+j}^i, this simplifies to

$$\begin{aligned}
p_t^i &= \frac{1}{f - gd_t} \sum_{j=1}^{\infty} \beta^j \left(f E_t \left(d_{t+j}^i \right) - g E_t \left(d_{t+j} d_{t+j}^i \right) \right) \\
&= \frac{1}{f - gd_t} \sum_{j=1}^{\infty} \beta^j \left(f E_t \left(\widehat{d}_{t+j} + \varepsilon_{t+j}^i \right) - g E_t \left(\left[\widehat{d}_{t+j} + \varepsilon_{t+j} \right] \left[\widehat{d}_{t+j} + \varepsilon_{t+j}^i \right] \right) \right) \\
&= \frac{1}{f - gd_t} \sum_{j=1}^{\infty} \beta^j \left(f E_t \left(\widehat{d}_{t+j} \right) - g E_t \left(\widehat{d}_{t+j}^2 + \frac{1}{2} \sigma^2 \right) \right),
\end{aligned}$$

so that $p_t^a = p_t^b$.

(d) Since a mutual fund is just the combination of 1/2 share of apple stock and 1/2 share of banana shock, it follows from arbitrage arguments that

$$p_t = \frac{1}{2} \left[p_t^a + p_t^b \right] = p_t^a = p_t^b.$$

(e) The realized rate of return is given by

$$R_t^i = \frac{1}{p_t}\left[p_{t+1} + \widehat{d}_{t+1} + \varepsilon_{t+1}^i\right],$$

with $\varepsilon_{t+1}^i = \varepsilon_{t+1}^a$ for apple stocks, ε_{t+1}^b for banana stocks, and ε_{t+1} for the mutual fund. It immediately follows that

$$E_t\left(R_t^i\right) = \frac{1}{p_t}E_t\left(p_{t+1} + \widehat{d}_{t+1}\right),$$

so that all three assets offer the same expected return. The mutual fund has safer dividends, but it does not have a higher price, or offer a lower expected return. This is because in equilibrium agents hold diversified asset portfolios, and are thus concerned only about how an assets returns covary with aggregate returns (consumption). (Recall that equity premia are based on $cov_t\left(R_t^i, u'\left(c_{t+1}\right)\right)$.) By this measure the mutual fund is no less risky than the individual stocks. Put differently, risks that can be diversified away do not enter into stock prices.

2. The main equations of our ad hoc model are:

$$y_t = p_t - p_t^e, \tag{AS}$$
$$y_t = m_t - p_t + \eta_t. \tag{AD}$$

Regardless of how it is constructed, p_t^e cannot use information outside the information set

$$I_{t-1} = \left\{y_{t-j}, p_{t-j}, m_{t-j}, \pi_{t-j}, p_{t-j}^e, \eta_{t-j}\right\}_{j=1}^{\infty}.$$

(a) Combining equations (AD) and (AS) yields

$$p_t - p_t^e = m_t - p_t + \eta_t,$$
$$p_t = \frac{1}{2}\left[m_t + \eta_t + p_t^e\right], \tag{EQP}$$
$$y_t = \frac{1}{2}\left[m_t + \eta_t - p_t^e\right]. \tag{EQY}$$

(b) We now assume that agents have adaptive expectations

$$p_t^e = \phi p_{t-1},$$

and that the money supply rule is

$$\begin{aligned} m_t &= \gamma p_{t-1}, \\ \gamma &= \arg\min_{\gamma_0} \alpha V\left(y_t\right) + V\left(p_t\right). \end{aligned}$$

1. Taking γ as given, we can simplify (EQP) and (EQY) to get

$$\begin{aligned} p_t &= \frac{1}{2}\left[\eta_t + \left(\gamma + \phi\right)p_{t-1}\right], \\ y_t &= \frac{1}{2}\left[\eta_t + \left(\gamma - \phi\right)p_{t-1}\right], \end{aligned}$$

2. The central bank's objective is

$$\min_{\gamma_0} \alpha\frac{1}{4}\left[\sigma_\eta^2 + (\gamma-\phi)^2 V(p_{t-1})\right] + \frac{1}{4}\left[\sigma_\eta^2 + (\gamma+\phi)^2 V(p_{t-1})\right].$$

This follows from the fact that $p_t^e = \phi p_{t-1}$ is independent of η_t, as the latter variable is not known at time $t-1$. Taking first order conditions, we get

$$\alpha\frac{2}{4}(\gamma-\phi)V(p_{t-1}) + \frac{2}{4}(\gamma+\phi)V(p_{t-1}) = 0,$$

or

$$\gamma = \phi\frac{\alpha-1}{1+\alpha}.$$

The second derivative is $\frac{1}{2}(\alpha+1)V(p_{t-1}) \geq 0$, confirming a minimum.

3. When $\alpha = 0$, the central bank places no weight on controlling output fluctuations, and $\gamma = -\phi$. Recalling our answer to part (i) above, we see that when $\alpha = 0$, $p_t = \frac{1}{2}\eta_t$ and the central bank offsets all price fluctuations known at time $t-1$. As $\alpha \to \infty$, the central bank places all of its weight on controlling output fluctuations and $\gamma \to \phi$. Recalling our answer to part (i) above, we see that as $\alpha \to \infty$, $y_t \to \frac{1}{2}\eta_t$ and the central bank offsets all output fluctuations known at time $t-1$. $\alpha = 1$ is the intermediate case, and γ takes on the intermediate value of 0.

(c) Now we assume that agents have rational expectations

$$p_t^e = E(p_t|\, I_{t-1}),$$

and the monetary rule is as before.

1. Taking γ as given, apply the law of iterated expectations to equation (EQP):

$$\begin{aligned} E(p_t|\, I_{t-1}) &= \frac{1}{2}E(\gamma p_{t-1} + \eta_t + p_t|\, I_{t-1}) = \gamma p_{t-1}, \\ p_t &= \frac{1}{2}\eta_t + \gamma p_{t-1}. \end{aligned}$$

Inserting this into equation (EQY) yields

$$y_t = \frac{1}{2}\left[\gamma p_{t-1} + \eta_t - \gamma p_{t-1}\right] = \frac{1}{2}\eta_t.$$

2. The central bank's objective is

$$\min_{\gamma_0} \alpha\frac{1}{4}\left[\sigma_\eta^2\right] + \frac{1}{4}\sigma_\eta^2 + \gamma^2 V(p_{t-1}).$$

This is minimized with $\gamma = 0$.

3. Under rational expectations monetary policy does not affect output, as it is predicted perfectly. The only remaining task for the central bank is price stability. Recalling our answer to part (1) above, we see that when $\gamma = 0$, $p_t = \frac{1}{2}\eta_t$ and prices depend only on variables not known at time $t-1$.

(d) In the setup used here, "leaning against the wind" (in terms of inflation) makes the most sense when agents have adaptive expectations and the central bank is concerned about price stability; $\gamma = -\phi$ under these circumstances. Under adaptive expectations, high realizations of current inflation (prices) lead agents to expect higher inflation in the future, putting upward pressure on future prices. Since p^e_{t+1} does not depend on m_{t+1}, a central bank can then use a decrease in m_{t+1} to offset this price pressure. Under rational expectations, high realizations of current inflation do not increase expected inflation *unless* the central bank responds with an increased money supply. In such circumstances, the best thing for the central bank to do is to keep the money supply stable.