Economics 701: Macroeconomics II
Spring 2005
Professor Jones

Final Examination
Answer Key

We are considering a version of the stochastic growth model with non-separable preferences. There is a representative price-taking firm that solves

$$\max_{L_t \geq 0} \Pi_t = Y_t - W_t L_t,$$

$$Y_t = Z_t L_t^{1-\alpha}, \quad 0 \leq \alpha < 1. \tag{PRF}$$

Ownership of the firm resides in stocks. Each period, a share entitles its owner to a share of the firm's profits—the firm retains no earnings—and we will normalize the number of shares to 1.

The household's preferences over consumption and labor are

$$E_0 \left(\sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma} \left(C_t^{\nu} \left(1 - L_t\right)^{1-\nu} \right)^{1-\gamma} \right).$$

Households receive labor income and profits from firms. Households earn a gross return of $(1+r) K_t$ on their assets, with $\beta(1+r) = 1$.

The log of technology, Z_t, follows an exogenous AR(1) process:

$$\widehat{z}_t \equiv \ln(Z_t) = \phi \widehat{z}_{t-1} + \varepsilon_t, \quad 0 \leq \phi < 1. \tag{TS}$$

where $\{\varepsilon_t\}$ is an exogenous stationary martingale difference sequence.

1. (4 points.) The firm's problem can be written as

$$\max_{L_t \geq 0} Z_t L_t^{1-\alpha} - W_t L_t,$$

and the order condition for profit maximization is

$$(1-\alpha) Z_t L_t^{-\alpha} = W_t. \tag{PM}$$

2. (14 points.) The Bellman equation for the household can be written as

$$\begin{aligned} V(K_t, Z_t) \quad = \quad & \max_{C_t \geq 0, L_t \in [0,1]} \frac{1}{1-\gamma} \left(C_t^{\nu} \left(1-L_t\right)^{1-\nu} \right)^{1-\gamma} \\ & + \beta E \left(V\left((1+r) K_t + W_t L_t + \Pi_t - C_t, Z_{t+1} \right) \middle| Z_t \right) \end{aligned}$$

The first order conditions are

$$\begin{aligned} \nu C_t^{\nu(1-\gamma)-1} \left(1-L_t\right)^{(1-\nu)(1-\gamma)} &= \beta E_t \left(\frac{\partial V[t+1]}{\partial K_{t+1}} \right), \\ (1-\nu) C_t^{\nu(1-\gamma)} \left(1-L_t\right)^{(1-\nu)(1-\gamma)-1} &= \beta E_t \left(\frac{\partial V[t+1]}{\partial K_{t+1}} \right) W_t. \end{aligned}$$

Because (following Benveniste-Scheinkman)

$$\begin{aligned}\frac{\partial V[t]}{\partial K_t} &= \beta E_t\left(\frac{\partial V[t+1]}{\partial K_{t+1}}\right)(1+r)\\ &= \nu C_t^{\nu(1-\gamma)-1}(1-L_t)^{(1-\nu)(1-\gamma)}(1+r),\end{aligned}$$

the first order conditions reduce to

$$\begin{aligned}\nu C_t^{\nu(1-\gamma)-1}(1-L_t)^{(1-\nu)(1-\gamma)}W_t &= (1-\nu)C_t^{\nu(1-\gamma)}(1-L_t)^{(1-\nu)(1-\gamma)-1},\\ \Rightarrow \nu\frac{W_t}{C_t} &= (1-\nu)\frac{1}{1-L_t},\\ C_t^{\nu(1-\gamma)-1}(1-L_t)^{(1-\nu)(1-\gamma)} &= E_t\left(C_{t+1}^{\nu(1-\gamma)-1}(1-L_{t+1})^{(1-\nu)(1-\gamma)}\right). \quad \text{(EE)}\end{aligned}$$

3. (5 points.) Along a balanced growth path, consumption and wages will both grow at the rate G, so that the ratio C_t/W_t will be constant. It then follows from equation (LL) that $L_t = 1 - \left(\frac{1-\nu}{\nu}\right)\left(\frac{C_t}{W_t}\right)$ will be constant as well.

4. (12 points.) If the derivative $-\frac{\partial^2 U(C,L)}{\partial C \partial L}$ is positive, consumption and **leisure** are complements.

   (a) The cross-partial derivative is

   $$\begin{aligned}-\frac{\partial^2 U(C,L)}{\partial C\partial L} &= -\frac{\partial}{\partial L}\nu C_t^{\nu(1-\gamma)-1}(1-L_t)^{(1-\nu)(1-\gamma)}\\ &= -(-1)\nu(1-\nu)(1-\gamma)C_t^{\nu(1-\gamma)-1}(1-L_t)^{(1-\nu)(1-\gamma)-1},\end{aligned}$$

   the sign of which depends on $\nu(1-\nu)(1-\gamma)$. When $\gamma > 1$, this term is negative, and consumption and leisure are substitutes.

   (b) Consumption and leisure will be substitutes when leisure-time activities such as cooking or cleaning (i.e., home production) substitute for market goods such as restaurant meals or made services.

   (c) Consumption and leisure will be complements when leisure-time activities such as skiing or painting lead the consumer to purchase additional goods and services. A person working 80 hours a week probably does not purchase many art supplies.

5. (9 points.) The Euler equation for capital was given by equation (EE) above. Using equation (PM) to eliminate wages, the labor allocation condition becomes

$$\nu(1-\alpha)\frac{1}{C_t}Z_t L_t^{-\alpha} = (1-\nu)\frac{1}{1-L_t}, \quad \text{(LL)}$$

The capital accumulation equation can be written as

$$K_{t+1} = (1+r)K_t + Y_t - C_t, \quad \text{(CA)}$$

with Y_t following equation (PRF). This constraint can either be derived directly as a resource constraint, or found by combining the household's budget constraint with the definition of profits.

6. (10 points.) Logging both sides of equation (LL) yields

$$\ln\left(\nu\frac{1-\alpha}{1-\nu}\right)-\ln C_t+\ln Z_t-\alpha\ln L_t=-\ln\left(1-\exp\left(\ln L_t\right)\right).$$

Implicitly differentiating this equation yields

$$-d\ln C_t-d\ln Z_t=\alpha d\ln L_t+\frac{\exp\left(\ln L_t\right)}{1-\exp\left(\ln L_t\right)}d\ln L_t.$$

Let lower-case letters with carats "$\widehat{\ }$" denote deviations of logged variables around their steady state values. The previous equation implies that

$$\begin{aligned}\widehat{\ell}_t &\approx \theta\left[\widehat{z}_t-\widehat{c}_t\right], &\text{(LL}'\text{)}\\ \theta &\equiv \left(\alpha+\frac{L_{SS}}{1-L_{SS}}\right)^{-1}.\end{aligned}$$

Proceeding similarly, it follows from equation (PRF) that

$$\begin{aligned}\widehat{y}_t &= \widehat{z}_t+(1-\alpha)\,\widehat{\ell}_t\\ &\approx \widehat{z}_t+(1-\alpha)\,\theta\left[\widehat{z}_t-\widehat{c}_t\right]\\ &= (1+\lambda)\,\widehat{z}_t-\lambda\widehat{c}_t, &\text{(PRF}'\text{)}\\ \lambda &= (1-\alpha)\,\theta.\end{aligned}$$

7. Let's consider the Euler equation.

   (a) Logging both sides of equation (EE) and imposing $\ln\left(E_t\left(X_t\right)\right)\approx E_t\left(\ln\left(X_t\right)\right)$ yields

$$\begin{aligned}&\left[\nu\left(1-\gamma\right)-1\right]\ln C_t+\left(1-\nu\right)\left(1-\gamma\right)\ln\left(1-\exp\left(\ln L_t\right)\right)\approx\\ &\quad E_t\left(\left[\nu\left(1-\gamma\right)-1\right]\ln C_{t+1}+\left(1-\nu\right)\left(1-\gamma\right)\ln\left(1-\exp\left(\ln L_{t+1}\right)\right)\right).\end{aligned}$$

   Implicitly differentiating this expression, we get

$$\begin{aligned}&\left[\nu\left(1-\gamma\right)-1\right]\left[d\ln C_t-E_t\left(d\ln C_{t+1}\right)\right]\approx\\ &\quad\left(1-\nu\right)\left(1-\gamma\right)\left[\frac{L_t}{1-L_t}d\ln L_t-\frac{L_{t+1}}{1-L_{t+1}}E_t\left(d\ln L_{t+1}\right)\right].\end{aligned}$$

   so that

$$\begin{aligned}\left[\nu\left(1-\gamma\right)-1\right]\left[\widehat{c}_t-E_t\left(\widehat{c}_{t+1}\right)\right] &\approx \left(1-\nu\right)\left(1-\gamma\right)\frac{L_{SS}}{1-L_{SS}}\left[\widehat{\ell}_t-E_t\left(\widehat{\ell}_{t+1}\right)\right]\\ &\approx \left(1-\nu\right)\left(1-\gamma\right)\frac{L_{SS}}{1-L_{SS}}\theta\left[\widehat{z}_t-E_t\left(\widehat{z}_{t+1}\right)-\widehat{c}_t+E_t\left(\widehat{c}_{t+1}\right)\right],\end{aligned}$$

   with the second line coming from equation (LL$'$). Collecting terms, we get

$$\begin{aligned}\widehat{c}_t-E_t\left(\widehat{c}_{t+1}\right) &\approx -\omega_3\left[\widehat{z}_t-E_t\left(\widehat{z}_{t+1}\right)\right]\\ \omega_3 &\equiv -\frac{\left(1-\nu\right)\left(1-\gamma\right)\frac{L_{SS}}{1-L_{SS}}\theta}{\nu\left(1-\gamma\right)-1+\left(1-\nu\right)\left(1-\gamma\right)\frac{L_{SS}}{1-L_{SS}}\theta},\end{aligned}$$

   and it then follows from equation (TS) that

$$\widehat{c}_t+\omega_3\left(1-\phi\right)\widehat{z}_t\approx E_t\left(\widehat{c}_{t+1}\right). \qquad \text{(EE}'\text{)}$$

(b) When $\gamma > 1$, it is straightforward to see that $(1-\nu)(1-\gamma)\frac{L_{SS}}{1-L_{SS}}\theta$ and $\nu(1-\gamma)$ are both negative. It then follows from our answer to part (a) that ω_3 is negative. (It can also be shown that $\omega_3 > -1$.) This means that a positive value of $\widehat{z}_t$ will increase current consumption, $\widehat{c}_t$, relative to future consumption, $E_t(\widehat{c}_{t+1})$. This is because an increase in $\widehat{z}_t$, by increasing the demand for labor will increase equilibrium wages and labor hours. With $\gamma > 1$, leisure and consumption are substitutes and hence $\widehat{c}_t$ will increase. Since technology shocks are persistent, future labor and thus $E_t(\widehat{c}_{t+1})$ will increase as well. But with $\phi < 1$, the net effect, given by $\omega_3(1-\phi)$, will be a relative increase in current consumption.

8. (26 points.) Proceeding in the usual fashion (see, e.g., homework 9) shows that the log-linear approximation of equation (CA) is:

$$\begin{aligned} \widehat{k}_{t+1} &= (1+r)\widehat{k}_t + \omega_1\widehat{z}_t - \omega_2\widehat{c}_t, \qquad \text{(CA}'\text{)} \\ \omega_1 &= (\psi - r)(1+\lambda) > 0, \\ \omega_2 &= \lambda(\psi - r) + \psi \\ &= \omega_1 + r. \end{aligned}$$

To solve the system given by equations (CA′), (EE′) and (TS), we begin by assuming that consumption can be written as a function of capital and preferences:

$$\widehat{c}_t = \eta\widehat{k}_t + \mu\widehat{z}_t. \qquad \text{(CF)}$$

(a) Combining equations (CF) and equation (CA′) yields

$$\begin{aligned} \widehat{k}_{t+1} &= (1+r-\omega_2\eta)\widehat{k}_t + (\omega_1 - \omega_2\mu)\widehat{z}_t \qquad \text{(CA}''\text{)} \\ &\equiv \pi_1\widehat{k}_t + \pi_2\widehat{z}_t. \end{aligned}$$

(b) Inserting equation (CF) into the *right*-hand-side of equation (EE′) yields

$$E_t(\widehat{c}_{t+1}) = \eta\widehat{k}_t + \mu\widehat{z}_t + \omega_3(1-\phi)\widehat{z}_t.$$

(c) Inserting equation (CF) into the *left*-hand-side of equation (EE′) yields

$$E_t(\widehat{c}_{t+1}) = E_t\left(\eta\widehat{k}_{t+1} + \mu\widehat{z}_{t+1}\right).$$

Inserting (CA″) and (TS) into this expression yields

$$\begin{aligned} E_t(\widehat{c}_{t+1}) &= \eta\left[(1+r-\omega_2\eta)\widehat{k}_t + (\omega_1-\omega_2\mu)\widehat{z}_t\right] + \mu\phi\widehat{z}_t, \\ &= \eta(1+r-\omega_2\eta)\widehat{k}_t + [\eta(\omega_1-\omega_2\mu) + \mu\phi]\widehat{z}_t. \end{aligned}$$

(d) If the expressions for $E_t(\widehat{c}_{t+1})$ in parts (b) and (c) are to be equal,

$$\begin{aligned} \eta(1+r-\omega_2\eta) &= \eta, \\ \eta(\omega_1-\omega_2\mu) + \mu\phi &= \mu + \omega_3(1-\phi). \end{aligned}$$

The first of these two equations has the non-zero solution

$$\eta = \frac{r}{\omega_2},$$

while the second equation implies that

$$\begin{aligned}\eta\omega_1 - \omega_3(1-\phi) &= \mu(1-\phi+\eta\omega_2) \\ &= \mu(1-\phi+r),\end{aligned}$$

so that

$$\mu = \frac{\eta\omega_1 - \omega_3(1-\phi)}{1-\phi+r}.$$

(e) Recall from question 7(b) that when $\gamma > 1$ (consumption and leisure are substitutes), ω_3 is negative. It then follows from $\omega_2 > \omega_1 > 0$ and $\phi < 1$ that η and μ are positive. This means that an increase in productivity will lead to an increase in consumption. Since higher productivity also leads to higher output—which you can assume—it follows that consumption and output will be positively correlated when when $\gamma > 1$. This is in fact what the data show. With separable preferences, and increase in z_t increases consumption because it increases lifetime wealth. Allowing consumption and leisure to be substitutes reinforces this effect; because an increase in z_t will raise the real wage, causing agents to substitute consumption for leisure.

As unrequested background, let's verify our assumption that output increases in $\widehat{z}_t$ when $\gamma > 1$. Note that $0 < \eta\omega_1 < r$. In addition, recall from question 7(b) that $\gamma > 1$ implies that $-1 < \omega_3 < 0$, so that $0 < -\omega_3(1-\phi) < 1-\phi$. We thus have

$$\mu < \frac{1-\phi+r}{1-\phi+r} = 1.$$

Finally, equation (PRF′) shows that

$$\begin{aligned}\widehat{y}_t &= \widehat{z}_t + \lambda\left[\widehat{z}_t - \widehat{c}_t\right] \\ &= \widehat{z}_t + \lambda\left[\widehat{z}_t - \eta\widehat{k}_t - \mu\widehat{z}_t\right] \\ &= \left[1+\lambda(1-\mu)\right]\widehat{z}_t - \eta\lambda\widehat{k}_t.\end{aligned}$$

With $\lambda > 0$ and $\mu < 1$, $\widehat{y}_t$ is unambiguously increasing in $\widehat{z}_t$.