Economics 701: Macroeconomics II

Spring 2009

Lecture 5: Real Business Cycles

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March, 2009

5. The Baseline RBC model and its Numerical Solution

(a) The model

- Ramsey model with variable labor supply and stochastic technology:

$$
\begin{aligned}
Y_t &= F\left(K_t^P, A_t L_t^P\right) = \left(K_t^P\right)^{\alpha}\left(A_t L_t^P\right)^{1-\alpha}, \\
A_t &= A_0 A^t \exp\left(z_t\right), \\
z_t &= \rho z_{t-1} + \varepsilon_t, \quad 0 \leq \rho < 1, \quad E_t\left(\varepsilon_{t+1}\right) = 0.
\end{aligned}
$$

- Assume $N_t = N_0 N^t$. Define:

$$
y_t = \frac{Y_t}{A_0 A^t N_t}, \quad k_t = \frac{K_t}{A_0 A^t N_t}, \quad c_t = \frac{C_t}{A_0 A^t N_t},
$$

$$
\ell_t = \frac{L_t}{N_t}, \quad w_t = \frac{W_t}{A_0 A^t}.
$$

5. The Baseline RBC model

(a) The model

- Producers operate under perfect competition, solve

$$\max_{k_t^P \geq 0, \ell_t^P \geq 0} e^{(1-\alpha)z_t} \left(k_t^P\right)^{\alpha} \left(\ell_t^P\right)^{1-\alpha} - r_t k_t^P - w_t \ell_t^P.$$

- The FOC are

$$\begin{aligned} r_t &= \frac{\partial F\left(k_t^P, \ell_t^P\right)}{\partial k_t^P} = \alpha \frac{y_t^P}{k_t^P}, \\ w_t &= \frac{\partial F\left(k_t^P, \ell_t^P\right)}{\partial \ell_t^P} = (1-\alpha) \frac{y_t^P}{\ell_t^P}. \end{aligned}$$

5. The Baseline RBC model

(a) The model

- The consumer's problem is:

$$\max_{\{c_t, k_{t+1}, \ell_t\}_{t=0}^{\infty}} E_0 \left(\sum_{t=0}^{\infty} (\beta N)^t \left[\ln\left(A_0 A^t c_t\right) + \chi \frac{(1-\ell_t)^{1-\gamma}}{1-\gamma} \right] \right),$$

$$s.t. \quad c_t + ANk_{t+1} = (1 + r_t - \delta)k_t + w_t \ell_t,$$

$$\ell_t \in [0,1], \qquad \text{(TE)}$$

$$\lim_{J\to\infty} \left(\prod_{j=1}^{J-1} [1 + r_{t+j} - \delta]^{-1} \right) A^{t+J} N^{t+J} k_{t+J} = 0.$$

and the other usual constraints.

5. (a) The model

- The first order conditions for the consumer's problem are:

$$\frac{1}{c_t} = \beta A^{-1} E_t \left(\frac{1}{c_{t+1}} [1 + r_{t+1} - \delta] \right),$$

$$\frac{1}{c_t} w_t = \chi (1 - \ell_t)^{-\gamma}.$$

- The resource constraint is:

$$\begin{aligned} y_t &= F\left(k_t^P, \ell_t^P\right) \\ &= c_t + ANk_{t+1} - (1-\delta)k_t. \end{aligned} \qquad \text{(RC)}$$

- We also need

$$c_t \geq 0, \quad k_t \geq 0. \qquad \text{(NNG)}$$

5. (a) The Model

- Given the initial stock k_0 and the stochastic process $\{A_t\}_{t=0}^{\infty}$, an equilibrium consists of the stochastic processes $\{c_t, k_{t+1}, \ell_t, r_t, w_t\}_{t=0}^{\infty}$ such that:
  - Given the process for prices $\{r_t, w_t\}$, $\{c_t, \ell_t, k_{t+1}\}$ solves the consumer's problem, and $\left\{k_t^P = k_t, \ell_t^P = \ell_t\right\}$ solves the producer's problem.
  - Markets clear: (RC), (TE) and (NNG) are satisfied.
- Comment: The market clearing conditions are redundant.

5. (a) In equilibrium, the economy follows

$$\frac{1}{c_t} = \beta A^{-1} E_t \left(\frac{1}{c_{t+1}} \left[\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right] \right), \qquad \text{(EE)}$$

$$\frac{1}{c_t} (1-\alpha) \frac{y_t}{\ell_t} = \chi (1 - \ell_t)^{-\gamma}, \qquad \text{(LL)}$$

$$y_t = e^{(1-\alpha) z_t} k_t^{\alpha} \ell_t^{1-\alpha}, \qquad \text{(PRF)}$$

$$A N k_{t+1} = y_t + (1-\delta) k_t - c_t, \qquad \text{(CA)}$$

$$z_t = \rho z_{t-1} + \varepsilon_t, \qquad \text{(TS)}$$

$$k_0 \text{ given}, \qquad \text{(K0)}$$

$$c_t \geq 0, \quad k_t \geq 0, \quad \ell_t \in [0,1], \qquad \text{(NNG)}$$

$$\lim_{J \to \infty} E_t \left((\beta N)^J \frac{k_{t+J+1}}{c_{t+J}} \right) = 0. \qquad \text{(TVC)}$$

5. The Baseline RBC model

(b) Calibration

- Parameters set to match other studies—sometimes micro-level—or to replicate steady-state ratios.
- $\alpha \in [0.3, 0.4]$: capital's share of national income.
- χ set so that $\ell_{ss} \in [0.2, 1/3]$: waking time allocated to work.
- $N \in [1.003, 1.004]$: quarterly population growth rate.

5. The Baseline RBC model

(b) Calibration

- Technology process
  - First, construct the technology series

$$\ln(A_t) = \frac{1}{1-\alpha}\left[\ln\left(Y_t\right) - \alpha\ln\left(K_t\right) - (1-\alpha)\ln\left(L_t\right)\right].$$

  - $A \in [1.003, 1.005]$: quarterly gross growth rate of $A_t =$ average value of Solow residual.
  - The stochastic residual z_t is derived as

$$z_t = \ln\left(A_t\right) - \ln\left(A_0\right) - \ln\left(A\right)\cdot t.$$

  - $\rho \approx 0.95$, $\sigma_\varepsilon \in [0.005, 0.01]$: estimated from quarterly $\{z_t\}$.

5. The Baseline RBC model

(b) Calibration

- $\delta \in [0.012, 0.025]$: quarterly depreciation rate, either estimated directly or inferred from $(Y_t - C_t)/K_t$.
- $\beta \in \left[0.985, 1.03^{-\frac{1}{4}}\right]$: satisfies

$$\beta A^{-1} R = \beta A^{-1} \left[\alpha \left(Y/K\right) + 1 - \delta\right] = 1.$$

- Comment: sometimes R is taken as the risk-free rate of return, in which case α and δ cannot be set independently.
- γ: wide variety of values.

5. The Baseline RBC model

(c) Solution method

- Log-linearization around a steady-state
- Let "$\widehat{y}$" denote logged deviations from the steady state:

$$\widehat{y}_t = \ln\left(\frac{y_t}{y_{ss}}\right) = \ln\left(1 + \frac{y_t - y_{ss}}{y_{ss}}\right)$$

$$\approx \frac{y_t - y_{ss}}{y_{ss}},$$

$$\widehat{k}_t = \ln\left(\frac{k_t}{k_{ss}}\right), \quad \widehat{c}_t = \ln\left(\frac{c_t}{c_{ss}}\right),$$

$$\widehat{\ell}_t = \ln\left(\frac{\ell_t}{\ell_{ss}}\right), \quad \widehat{z}_t = z_t.$$

5. The Baseline RBC model

(c) Solution method

- Log-linearized production function (PRF):

$$\widehat{y}_t = \ln\left(\frac{e^{(1-\alpha)z_t}k_t^{\alpha}\ell_t^{1-\alpha}}{k_{ss}^{\alpha}\ell_{ss}^{1-\alpha}}\right)$$

$$= (1-\alpha)\,z_t + \alpha\widehat{k}_t + (1-\alpha)\,\widehat{\ell}_t. \qquad \text{(PRF}'\text{)}$$

- Log-linearized labor-leisure trade-off (LL):

$$(1-\alpha)\,\frac{y_t}{c_t\ell_t} = (1-\alpha)\exp\left(\ln\left(y_t\right) - \ln\left(c_t\right) - \ln\left(\ell_t\right)\right)$$

$$= \chi\left(1-\exp\left(\ln\left(\ell_t\right)\right)\right)^{-\gamma},$$

5. (c) Solution method

- Log-linearized labor-leisure trade-off (LL) (continued):

$$\ln(1-\alpha)+\ln(y_t)-\ln(c_t)-\ln(\ell_t)=$$
$$\ln(\chi)-\gamma\ln(1-\exp(\ln(\ell_t))),$$
$$d\ln(y_t)-d\ln(c_t)-d\ln(\ell_t)=$$
$$\gamma\frac{1}{1-\exp(\ln(\ell_t))}\exp(\ln(\ell_t))\,d\ln(\ell_t).$$

Around a steady state:

$$\exp(\ln(\ell_t))\approx\ell_{ss},$$
$$d\ln(y_t)\approx\ln(y_t)-\ln(y_{ss})=\widehat{y}_t,\text{ etc.}$$

5. (c) Solution method

- Log-linearized labor-leisure trade-off (LL) (continued): We finally get

$$\widehat{y}_t - \widehat{c}_t \approx \left[1 + \gamma \frac{\ell_{ss}}{1-\ell_{ss}}\right] \widehat{\ell}_t$$
$$= \left[1 + \frac{1}{IES_L}\right] \widehat{\ell}_t. \qquad \text{(LL}'\text{)}$$

5. (c) Log-linearized capital accumulation equation:

$$AN \exp\left(\ln\left(k_{t+1}\right)\right) = \exp\left(\ln\left(y_t\right)\right) + (1-\delta)\exp\left(\ln\left(k_t\right)\right) - \exp\left(\ln\left(c_t\right)\right),$$

$$AN \exp\left(\ln\left(k_{t+1}\right)\right) d\ln\left(k_{t+1}\right) = (1-\delta)\exp\left(\ln\left(k_t\right)\right) d\ln\left(k_t\right) + \exp\left(\ln\left(y_t\right)\right) d\ln\left(y_t\right) - \exp\left(\ln\left(c_t\right)\right) d\ln\left(c_t\right).$$

This simplifies to

$$ANk_{ss}\widehat{k}_{t+1} \approx (1-\delta)\,k_{ss}\widehat{k}_t + y_{ss}\widehat{y}_t - c_{ss}\widehat{c}_t,$$

$$\widehat{k}_{t+1} \approx \frac{(1-\delta)}{AN}\widehat{k}_t + \left(\frac{1}{AN}\frac{y_{ss}}{k_{ss}}\right)\widehat{y}_t - \left(\frac{1}{AN}\frac{c_{ss}}{k_{ss}}\right)\widehat{c}_t. \qquad (\text{CA}')$$

5. (c) Solution method

- The Euler equation is log-linearized in a similar way, yielding equation (EE′).
- Vector LEDE
  - Use equations (PRF′) and (LL′) to eliminate output and labor from equations (CA′) and (EE′).
  - Rearrange and add equation (TS) to get

$$E_t \begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \\ z_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \\ z_t \end{pmatrix},$$

Impose

$$\begin{aligned} z_{t+1} - E_t\left(z_{t+1}\right) &= \varepsilon_{t+1}, \quad z_0 \text{ given}, \\ \widehat{k}_{t+1} - E_t\left(\widehat{k}_{t+1}\right) &= 0, \quad \widehat{k}_0 \text{ given}. \end{aligned}$$

5. (c) Solution method

- Saddle-path stability
  - **A** has two eigenvalues with modulus less than 1 and one eigenvalue of modulus greater than 1 $\Rightarrow$

$$
\begin{aligned}
\widehat{c}_t &= \mathbf{a}\begin{pmatrix} \widehat{k}_t \\ z_t \end{pmatrix}, \\
\Rightarrow \widehat{c}_{t+1} - E_t\left(\widehat{c}_{t+1}\right) &= \mathbf{a}\begin{pmatrix} \widehat{k}_{t+1} - E_t\left(\widehat{k}_{t+1}\right) \\ z_{t+1} - E_t\left(z_{t+1}\right) \end{pmatrix} \\
&= \begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{pmatrix} 0 \\ \varepsilon_{t+1} \end{pmatrix},
\end{aligned}
$$

5. (c) Solution method

- Saddle-path stability
  - $\mathbf{A}$ has two eigenvalues with modulus less than 1 and one eigenvalue of modulus greater than 1 $\Rightarrow$

$$\begin{pmatrix} \widehat{k}_{t+1} \\ \widehat{c}_{t+1} \\ z_{t+1} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \widehat{k}_{t} \\ \widehat{c}_{t} \\ z_{t} \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \varepsilon_{t+1} \\ \varepsilon_{t+1} \end{pmatrix}. \quad \text{(RBC)}$$

  - Analogous to saddle-path stability in deterministic Ramsey model, where there was one saddle-path value of c_t for each value of k_t.

5. (c) Solution method

- Other variables are approximated as linear functions of the core system:

$$\begin{pmatrix} \widehat{y}_t \\ \widehat{\ell}_t \end{pmatrix} = \mathbf{R} \begin{pmatrix} \widehat{k}_t \\ \widehat{c}_t \\ z_t \end{pmatrix}.$$

6. Analysis of the Baseline RBC Model

(a) The RBC interpretation of the IS-FE Model

- The IS Curve
  - Output-interest rate combinations where desired saving = desired investment
  - Desired saving:

$$s_t^d = y_t - c\left(y_t^P, \left(1 - \tau_{t+1}^k\right) r_{t+1}\right) - g_t.$$

6. (a) The RBC interpretation of the IS-FE Model

- The IS Curve
  - Notation:

$$
\begin{aligned}
y_t^P &= \text{after-tax permanent income}, \\
r_{t+1} &= \text{real interest rate}, \\
\widetilde{r}_{t+1} &= \text{rental price of capital} \\
&= r_{t+1} + \delta, \\
\delta &= \text{depreciation rate}, \\
\tau_{t+1}^k &= \text{marginal tax rate on interest income}, \\
g_t &= \text{government spending}.
\end{aligned}
$$

  - Pretend (for now) that $\widetilde{r}_{t+1}$ and τ_{t+1}^k are known.

6. (a) The RBC interpretation of the IS-FE Model

- The IS Curve
  - When future income is held fixed

$$\frac{\partial c}{\partial y_t} = \frac{\partial c}{\partial y_t^P}\frac{\partial y_t^P}{\partial y_t} \in (0, 1),$$

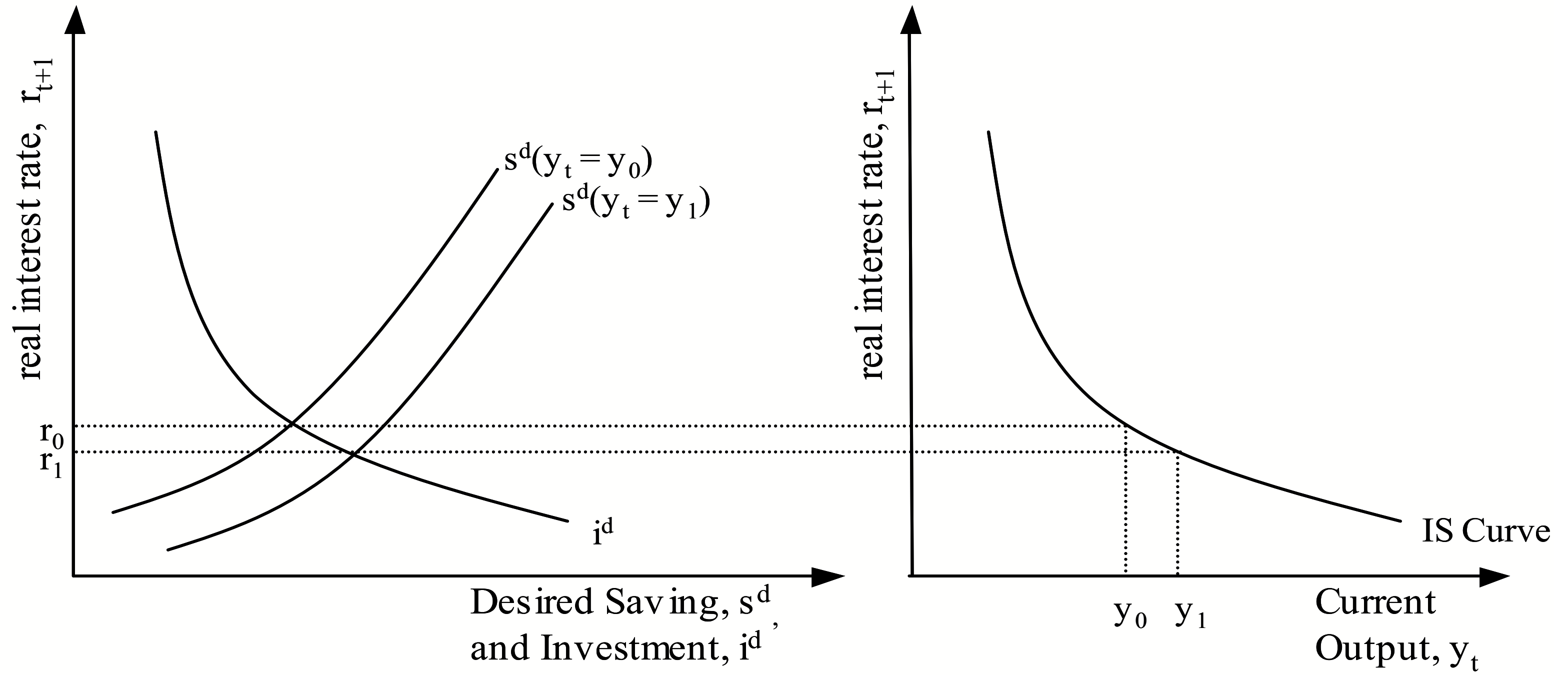
    but is generally much less than one.
  - Desired investment:

$$i_t^d = i\left(E_t\left(MPK_{t+1}\right), k_t, r_{t+1} + \delta\right).$$

  - Assume that corporate taxes are borne by household.
  - Equilibrium: $s_t^d = i_t^d$.

6. (a) The IS Curve

- Vary current income, y_t, to trace out the IS curve:



- Informal homework: show that anything that increases $\left(c_t^d + g_t + i_t^d\right)$, holding r_t and y_t fixed, shifts the IS curve to the right.
- Shifters include taxes, future income, future TFP.

6. (a) The RBC interpretation of the IS-FE Model

- Labor Markets
  - Frisch labor supply curve:

$$\ell_t^{sf} = \ell^{sf}\left(\left(1-\tau_t^{\ell}\right) w_t, MU_C\right),$$

where τ_t^{ℓ} is the marginal tax rate on labor income.

  - “Regular” labor supply curve

$$\begin{aligned} \ell_t^s &= \ell^s\left(\left(1-\tau_t^{\ell}\right) w_t, y_t^P\left(w_t, \tau_t^{\ell}\right)\right) \\ &= \ell^s\left(w_t, \tau_t^{\ell}\right), \end{aligned}$$

incorporates effects of w_t and τ_t^{ℓ} on permanent income.

6. (a) The RBC interpretation of the IS-FE Model

- Labor Markets
  - Labor demand curve:

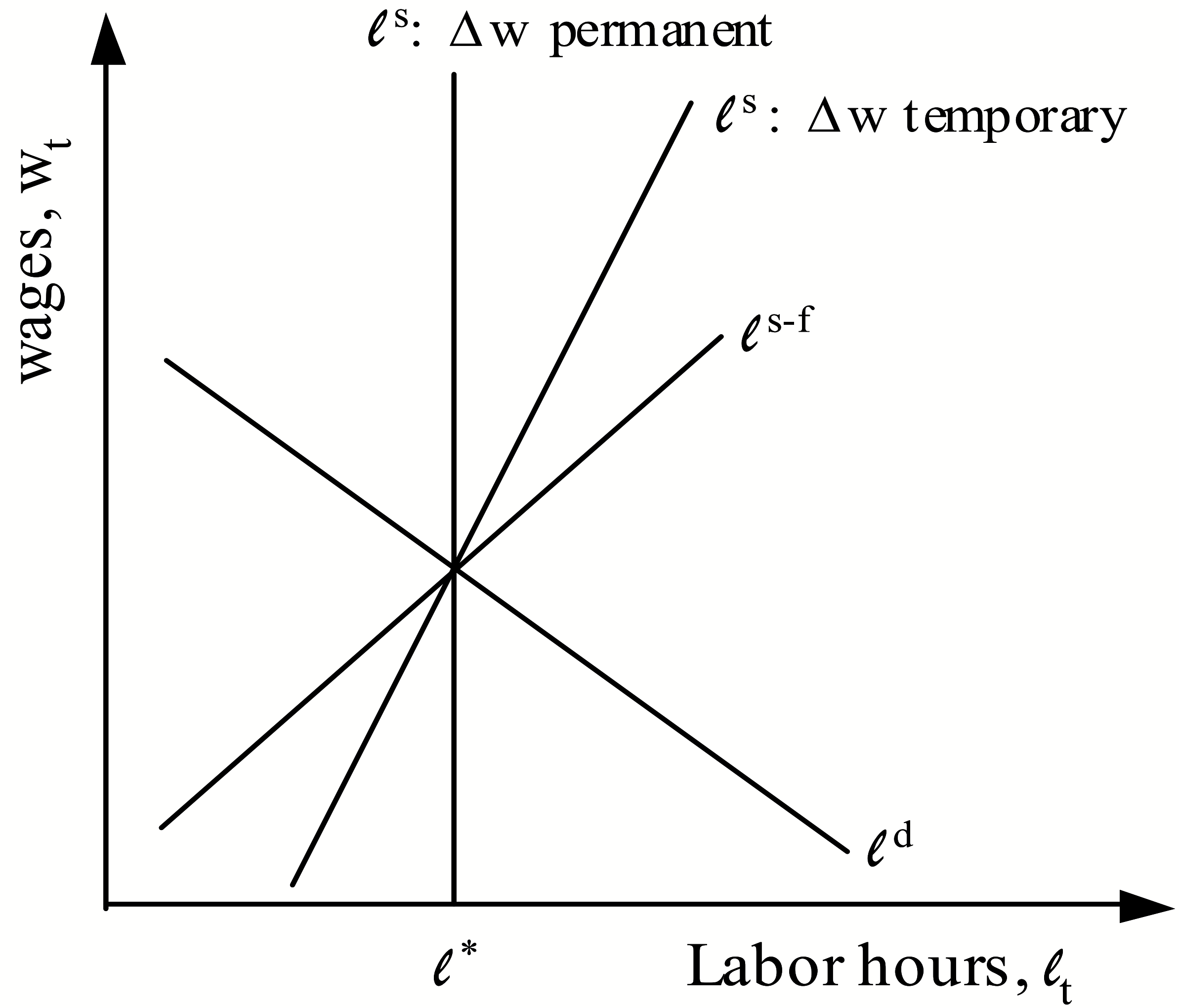
$$\ell_t^d = \ell^d \left(MPL_t, w_t \right).$$

  - Equilibrium:

$$\ell^d \left(MPL_t, w_t \right) = \ell^s \left(w_t, \tau_t^\ell \right).$$

6. (a) Labor Markets

- Equilibrium



6. (a) The RBC interpretation of the IS-FE Model

- Full-employment (output supply) line
  - Production function evaluated at equilibrium labor:

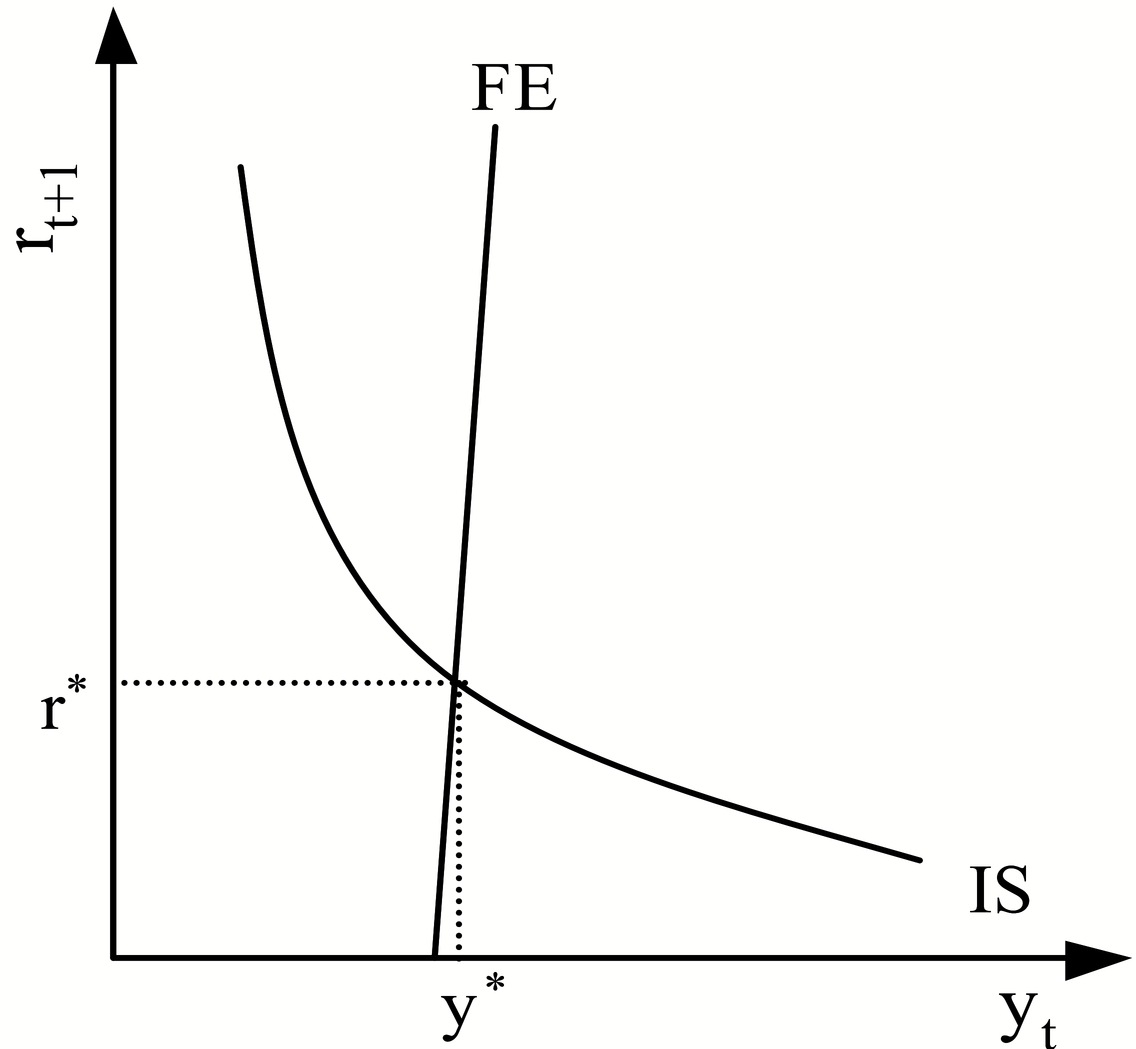
$$y_t = e^{(1-\alpha)z_t} k_t^{\alpha} \left(\ell_t^*\right)^{1-\alpha}.$$

  - FE slopes up in r because $r \uparrow \Rightarrow$ postpone leisure $\Rightarrow \ell_t^s$ shifts right $\Rightarrow \ell^* \uparrow$. Follows from

$$\chi \frac{(1-\ell_t)^{-\gamma}}{w_t} = \beta A^{-1} \chi E_t \left(\frac{(1-\ell_{t+1})^{-\gamma}}{w_{t+1}} R_{t+1} \right).$$

6. (a) The RBC interpretation of the IS-FE Model

- IS-FE Equilibrium



6. Analysis of the Baseline RBC Model

(b) Responses to a transitory but persistent technology shock

- Numerical methodology
  - Set $\widehat{k}_0 = \widehat{c}_0 = z_0 = 0$.
  - Set $\varepsilon_1 = 1$, $\varepsilon_2 = \varepsilon_3 = ... = 0$.
  - Follow dynamics of the linear system (RBC).
- Initial responses in the goods market
  - $E_t\left(MPK_{t+1}\right) \uparrow \Rightarrow i^d$ shifts right, future income$\uparrow \Rightarrow s^d$ shifts left $\Rightarrow$ IS curve shifts right.
  - $z \uparrow \Rightarrow$ FE shifts right as well. Ultimately, c, i, y and r increase.
  - $\Delta i/i > \Delta c/c$, as households use capital to smooth consumption.

6. (b) Responses to a transitory but persistent technology shock

- Initial responses in the labor market
  - $z \uparrow \Rightarrow \ell^d$ shifts right.
  - $y_t^P \uparrow \Rightarrow \ell^{sf}$ shifts in.
  - Ultimately ℓ and w increase.
- Longer-run dynamics
  - Technology: z gradually declines over time $\Rightarrow$ marginal products falling.
  - Capital: $i \uparrow \Rightarrow k \uparrow$; k increases, then returns to k_0 as $E_t\left(MPK_{t+1}\right)$ falls.
  - Output: y monotonically returns to y_0, pattern similar to that of z.

6. (b) Responses to a transitory but persistent technology shock

- Longer-run dynamics
  - Interest rates: $k \uparrow$, $z \downarrow \Rightarrow E_t\left(MPK_{t+1}\right) \downarrow \Rightarrow$ IS curve shifts left, while FE line shifts left to a lesser extent $\Rightarrow r$ falls.
  - Consumption: with r constant, permanent income theory $\Rightarrow$ permanent increase in c; but as r drops below r_0, agents reduce saving; c rises, then returns to c_0.
  - Labor: $z \downarrow \Rightarrow MPL \downarrow$ (outweighs effect of $k \uparrow$) and $r \downarrow$ (current MU_C down) $\Rightarrow \ell^d$ and ℓ^s both shift left $\Rightarrow w$ stays high, but ℓ falls below ℓ_0.

6. Analysis of the Baseline RBC Model

(c) Second Moments

- Unfiltered data
  - Use equation (RBC) to find second moments by solving

$$V\begin{pmatrix}\widehat{k}_{t+1}\\ \widehat{c}_{t+1}\\ z_{t+1}\end{pmatrix} = V\left(\mathbf{A}\begin{pmatrix}\widehat{k}_{t}\\ \widehat{c}_{t}\\ z_{t}\end{pmatrix}+\begin{pmatrix}0\\ a_2\varepsilon_{t+1}\\ \varepsilon_{t+1}\end{pmatrix}\right)$$

$$\Rightarrow V\begin{pmatrix}\widehat{k}\\ \widehat{c}\\ z\end{pmatrix} = \mathbf{A}V\begin{pmatrix}\widehat{k}\\ \widehat{c}\\ z\end{pmatrix}\mathbf{A}'+V\begin{pmatrix}0\\ a_2\varepsilon\\ \varepsilon\end{pmatrix}.$$

6. Analysis of the Baseline RBC Model

(c) Second Moments

- Hodrick-Prescott-filtered data
  - Find second moments by simulation
    i. Use a random number generator to create $\{\varepsilon_t\}_{t=0}^{T}$.
    ii. Feed k_0, z_0, $\{\varepsilon_t\}$ into equation (RBC) to generate artificial time series.
    iii. Run these artificial time series through the HP filter.
    iv. Find and save second moments for filtered, simulated data.
    v. Average across simulations.
  - $\exists$ complicated but quick analytical approximations.

6. Analysis of the Baseline RBC Model

(c) Second Moments

- Results
  - Consumption less volatile than output, investment more volatile.
  - All variables are very procyclical: reflects single shock.
  - If technology is uncorrelated, so are output and employment. Endogenous labor and capital accumulation do not generate powerful dynamics.