Economics 701: Macroeconomics II

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Lecture 4: Asset Pricing

Lag Operators and
Linear Expectational Difference Equations (LEDEs)

University at Albany
State University of New York

John Bailey Jones

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3. Lag Operators and Linear Expectational Difference Equations (LEDE)

(a) Lag operators

- Let **S** be a set of stochastic processes. Define the lag operator $L^n : \mathbf{S} \to \mathbf{S}$, n an integer, by

$$L^n \{X_t\}_{t=-\infty}^{\infty} = \{X_{t-n}\}_{t=-\infty}^{\infty} .$$

- The lag operator is linear:

$$L\,(aX_t + bL^n X_t) = (aL + bL^{n+1})X_t,$$

so that lag operations can be manipulated like polynomials.

3. (a) Lag operators

- Because the lag operator is linear,

$$(1-\phi L^n)\left(\sum_{j=0}^{J}(\phi L^n)^j\right)X_t = \left(1-(\phi L^n)^{J+1}\right)X_t,$$

and if $(\phi L^n)^{J+1}X_t$ and $\left(\sum_{j=0}^{J}(\phi L^n)^j\right)X_t$ "converge"—which might be true even if $|\phi|>1$—we get

$$\frac{1}{1-\phi L^n}X_t = \left(\sum_{j=0}^{\infty}(\phi L^n)^j\right)X_t,$$

the inverse of the operation $1-\phi L^n$.

3. (a) Lag operators

- Suppose $X_t = c$, $\forall t$. Then

$$L^n c = L^n X_t = c.$$

- Definition: the lag operator does **not** shift information sets:

$$L^n E_t \left(X_{t+j} \right) = E_t \left(X_{t+j-n} \right) \neq E_{t-n} \left(X_{t+j-n} \right).$$

3. (a) Lag operators

- **Warning:** Although

$$\begin{aligned} w_t = y_t &\Rightarrow E_t\left(w_t\right) = E_t\left(y_t\right) \\ &\Rightarrow L^{-1}E_t\left(w_t\right) = L^{-1}E_t\left(y_t\right) \\ &\Rightarrow E_t\left(w_{t+1}\right) = E_t\left(y_{t+1}\right), \end{aligned}$$

  the opposite does not hold, i.e,

$$\begin{aligned} E_t\left(w_{t+1}\right) = E_t\left(y_{t+1}\right) &\not\Rightarrow LE_t\left(w_{t+1}\right) = LE_t\left(y_{t+1}\right) \\ &\Rightarrow w_t = y_t. \end{aligned}$$

3. (b) Homogeneous first-order LEDE

- Basic equation

$$E_t\left(b_{t+1}\right) = \lambda b_t \Leftrightarrow E_t\left(\left(1-\lambda L\right) b_{t+1}\right) = 0.$$

  - Rewrite this as

$$\begin{aligned} b_{t+1} &= \lambda b_t + \varepsilon_{t+1}, \\ \varepsilon_{t+1} &\equiv b_{t+1} - E_t\left(b_{t+1}\right). \end{aligned}$$

  - As a forecast error, ε_t forms a martingale difference sequence, i.e.,

$$E_t\left(\varepsilon_{t+1}\right) = 0.$$

3. (b) Homogeneous first-order LEDE

- Basic equation
  - Generalize:

$$\begin{aligned} b_{t+1} - c\lambda^{t+1} &= \lambda b_t - \lambda c \lambda^t + \varepsilon_{t+1}, \\ (1 - \lambda L)\left(b_{t+1} - c\lambda^{t+1}\right) &= \varepsilon_{t+1}, \end{aligned}$$

    where c is a constant.
  - Goal: find the set of admissible $\{\varepsilon_t\}$ and c.

3. (b) Homogeneous first-order LEDE

- Backward solution.
  - If time starts at $-\infty$, the backward solution—if it is well-defined—is

$$b_t = c\lambda^t + \sum\nolimits_{j=0}^{\infty} \lambda^j \varepsilon_{t-j}.$$

  - If time starts at 0, the backward solution is

$$b_t = b_0\lambda^t + \sum\nolimits_{j=0}^{t-1} \lambda^j \varepsilon_{t-j},$$

where b_0 is a (possibly) random variable.

3. (b) Homogeneous first-order LEDE

- Restrictions to reduce the solution set
  - “Initial Conditions": $\{\varepsilon_t\}$ and b_0 are given. For example, under perfect foresight $\varepsilon_t = 0$, $\forall t$.
  - Non-explosiveness:

$$\lim_{j\to\infty} E_t\left(b_{t+j}\right) = 0, \quad \forall t,$$

$$\sup_t V\left(b_t\right) < \infty.$$

  - Note that

$$\begin{aligned} E_t\left(b_{t+2}\right) &= E_t\left(E_{t+1}\left(b_{t+2}\right)\right) \\ &= E_t\left(\lambda b_{t+1}\right) = \lambda\left(\lambda b_t\right), \\ \Rightarrow E_t\left(b_{t+j}\right) &= \lambda^j b_t. \end{aligned}$$

3. (b) Homogeneous first-order LEDE

- Restrictions to reduce the solution set
  - If $|\lambda| < 1$, there are many c and $\{\varepsilon_t\}$ where the non-explosiveness conditions do not restrict.
  - But if $|\lambda| \geq 1$, the only admissible solution is $\varepsilon_t = c = 0$, so that $b_t = 0, \forall t$.
  - Note that if $|\lambda| \geq 1$, b_t cannot satisfy both an "initial condition" and a non-explosiveness condition.

3. (c) General LEDE

- Basic equation
  - We wish to solve

$$E_t\left(x_{t+1}\right) = \lambda x_t + z_t,$$

where $\{z_t\}$ is a stochastic <u>forcing process</u>.

  - More generally

$$\begin{aligned} E_t\left(x_{t+1} - b_{t+1}\right) &= \lambda x_t + z_t - \lambda b_t \\ \Leftrightarrow E_t\left(\left(1 - \lambda L\right)\left(x_{t+1} - b_{t+1}\right)\right) &= z_t, \end{aligned}$$

where b_{t+1} is a “<u>bubble term</u>” that solves

$$E_t\left(b_{t+1}\right) = \lambda b_t.$$

3. (c) General LEDE

- Basic equation
  - The general problem is

$$
\begin{aligned}
x_{t+1} - b_{t+1} &= \lambda \left(x_t - b_t \right) + \widetilde{\eta}_{t+1} + z_t, \\
\widetilde{\eta}_{t+1} &\equiv \left(x_{t+1} - b_{t+1} \right) - E_t \left(x_{t+1} - b_{t+1} \right), \\
\left(1 - \lambda L \right) \left(x_{t+1} - b_{t+1} \right) &= \widetilde{\eta}_{t+1} + z_t. \qquad \text{(GP)}
\end{aligned}
$$

- Goal: find the set of admissable $\{\widetilde{\eta}_t\}$ and $\{b_t\}$.

3. (c) General LEDE

- Backward solution
  - $\{\widetilde{\eta}_t\}$ and $\{b_t\}$ cannot be identified separately.
  - If time starts at $-\infty$, the backward solution—if it is well-defined—is

$$
\begin{aligned}
x_{t+1} &= \sum_{j=0}^{\infty} \lambda^j \left(z_{t-j} + \widetilde{\eta}_{t+1-j}\right) + b_{t+1} \\
&= \sum_{j=0}^{\infty} \lambda^j z_{t-j} + \widetilde{b}_{t+1}, \\
\widetilde{b}_{t+1} &\equiv b_{t+1} + \sum_{j=0}^{\infty} \lambda^j \widetilde{\eta}_{t+1-j}.
\end{aligned}
$$

  - Informal homework: show that $\widetilde{b}_{t+1}$ is a bubble term.
  - $\sum_{j=0}^{\infty} \lambda^j z_{t-j}$ is sometimes called the fundamental, or particular solution.

3. (c) General LEDE

- Backward solution
  - If time starts at 0, the backward solution can be written as

$$\begin{aligned} x_{t+1} = \; & \sum_{j=0}^{t} \lambda^{j} z_{t-j} + \sum_{j=0}^{t} \lambda^{j} \widetilde{\eta}_{t+1-j} \\ & + \left(x_0 - b_0\right) \lambda^{t+1} + b_{t+1}, \end{aligned}$$

which becomes

$$\begin{aligned} x_{t+1} &= \sum_{j=0}^{t} \lambda^{j} z_{t-j} + \sum_{j=0}^{t} \lambda^{j} \eta_{t+1-j} + x_0 \lambda^{t+1}, \\ \eta_t &\equiv \widetilde{\eta}_t + b_t - E_{t-1}\left(b_t\right) \\ &= x_t - E_{t-1}\left(x_t\right). \end{aligned}$$

3. (c) General LEDE

- Forward solution
  - First, rewrite (GP) as:

$$\left(\frac{1-\lambda L}{-\lambda L}\right)(-\lambda L)\left(x_{t+1}-b_{t+1}\right)=\widetilde{\eta}_{t+1}+z_t,$$

$$\left(1-\lambda^{-1}L^{-1}\right)\left(x_t-b_t\right)=-\frac{1}{\lambda}\left(z_t+\widetilde{\eta}_{t+1}\right).$$

  - To ensure that x_t is a function only of variables known at time t, write this as

$$E_t\left(\left(1-\lambda^{-1}L^{-1}\right)\left(x_t-b_t\right)\right)=-\frac{1}{\lambda}E_t\left(z_t+\widetilde{\eta}_{t+1}\right).$$

3. (c) General LEDE

- Forward solution
  - Invert the lag operator:

$$E_t\left(x_t - b_t\right) = -\frac{1}{\lambda} E_t\left(\frac{1}{1-\lambda^{-1}L^{-1}}\left(z_t + \widetilde{\eta}_{t+1}\right)\right),$$

$$\begin{aligned} x_t &= -\frac{1}{\lambda} E_t\left(\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j \left(z_{t+j} + \widetilde{\eta}_{t+j+1}\right)\right) + b_t, \\ &= -\frac{1}{\lambda} E_t\left(\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right) + b_t, \end{aligned}$$

because $E_t\left(\widetilde{\eta}_{t+j}\right) = 0, \forall j \geq 1.$

3. (c) General LEDE

- Forward solution
  - $-\frac{1}{\lambda}E_t\left(\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right)$—if it is well-defined—is sometimes called a fundamental, or particular solution.
  - Note that $\widetilde{\eta}_t$ depends only on the forcing process z_t:

$$\widetilde{\eta}_t = -\frac{1}{\lambda}\left[E_t\left(\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right)\right.$$
$$\left. - E_{t-1}\left(\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^j z_{t+j}\right)\right], \forall t.$$

3. (c) General LEDE

- Summary
  - Forward solution

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right) + b_t.$$

  - Backward solution

$$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + b_{t+1},$$

or

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

3. (c) General LEDE

- Restrictions
  - "Initial Conditions:" x_0 <u>and</u> $\{\eta_t\}_{t=1}^{\infty}$ are directly given. Ex.: capital accumulation:

$$\begin{aligned} & k_{t+1} = (1-\delta)\,k_t + i_t, \\ & k_0 \text{ given}, \\ & k_{t+1} - E_t\left(k_{t+1}\right) = 0. \end{aligned}$$

  - Non-explosiveness:

$$\begin{aligned} & \lim_{j\to\infty} E_t\left(x_{t+j}\right) = 0, \quad \forall t, \\ & \sup_t V\left(x_t\right) < \infty. \end{aligned}$$

3. (c) General LEDE

- Restrictions
  - If $|\lambda| < 1$, for “well-behaved” $\{z_t\}$ (e.g, ARMA processes), one solves $(1 - \lambda L)^{-1}$ backwards to get
  $$x_{t+1} = \sum_{j=0}^{\infty} \lambda^j z_{t-j} + b_{t+1},$$
  with a large number of permissable $\{b_t\}$.
  - But if $|\lambda| > 1$, for “typical” $\{z_t\}$ (e.g, ARMA processes), we must solve $(1 - \lambda L)^{-1}$ forward and set $b_t = 0$, so that
  $$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda}\right)^j z_{t+j} \right).$$
  - If $|\lambda| > 1$, x_t cannot satisfy both “initial conditions” and non-explosiveness.

3. (d) Summary/Rule of thumb:

- If $|\lambda| < 1$, set

$$x_{t+1} = \sum_{j=0}^{t} \lambda^j z_{t-j} + \sum_{j=0}^{t} \lambda^j \eta_{t+1-j} + x_0 \lambda^{t+1}.$$

  and use “initial conditions” to pin down x_0 and $\{\eta_t\}$.

- If $|\lambda| > 1$, set

$$x_t = -\frac{1}{\lambda} E_t \left(\sum_{j=0}^{\infty} \left(\frac{1}{\lambda} \right)^j z_{t+j} \right).$$

- If $|\lambda| = 1$, consider case-by-case.