Economics 701: Macroeconomics II

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Lecture 3: Optimal Consumption Under Uncertainty

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1. Maximization under uncertainty: see technical notes.

2. The finite-horizon consumer's problem:

$$\max_{\{c_t\}_{t=0}^{T}} E_0 \left(\sum_{t=0}^{T} \beta^t u(c_t) \right)$$

$$s.t. \quad A_{t+1} = (1+r)(A_t + y_t - c_t), \quad t = 0, 1, ..., T, \qquad \text{(FBC)}$$

$$A_{T+1} \geq 0, \qquad \text{(NPG)}$$

$$c_t \geq 0, \quad t = 0, 1, ..., T, \qquad \text{(NNG)}$$

$$A_0 \text{ given}, \qquad \text{(A0)}$$

y_t = exogenous random income,

A_t = real value of risk-free asset,

c_t = consumption.

2. The finite-horizon consumer's problem (continued)

$$I_t = \text{information set} = \left\{ \{A_{t-j}, y_{t-j}\}_{j=0}^{\infty}, \{c_{t-j}\}_{j=1}^{\infty} \right\},$$

Note: A_{t+1} and c_t are also known,

as they are being picked,

$$c_t = c_t(I_t) = \text{contingency plan},$$

$$\beta \in (0, 1),$$

$$r > 0, \text{ with } \beta(1+r) \leq 1,$$

$$u'(\cdot) > 0, \quad u''(\cdot) < 0.$$

- (FBC) is the flow budget constraint.
- (NPG) is the No-Ponzi-Game condition; otherwise, borrow and consume infinitely.

2. • Write as a Lagrangian:

$$\mathcal{L} = E_0\Bigg(\sum\nolimits_{t=0}^{T} \beta^t \left[u\left(c_t\right) + \lambda_t \left(A_t + y_t - c_t - \frac{A_{t+1}}{1+r} \right) \right] + \beta^T \lambda_{T+1} A_{T+1} \Bigg).$$

• The FOC for an interior solution ($c_t > 0$) include

$$\begin{aligned} u'\left(c_t\right) &= \lambda_t, \quad t = 0, 1, ..., T, \\ \frac{\lambda_t}{1+r} &= \beta E_t\left(\lambda_{t+1}\right) \quad t = 0, 1, ..., T-1, \\ \lambda_{T+1} &= \frac{\lambda_T}{1+r}, \\ A_{T+1} &\geq 0; \quad \beta^T \lambda_{T+1} A_{T+1} = 0. \end{aligned}$$

2. - The FOC reduce to

$$u'(c_t) = \beta(1+r) E_t \left(u'(c_{t+1})\right), \quad t = 0, 1, ..., T-1, \tag{EE}$$

$$A_{T+1} = 0 \Leftrightarrow c_T = A_T + y_T, \tag{NPG$'$}$$

- (EE) and (NPG$'$) imply that

$$\beta^T E_0 \left\{u'(c_T) A_{T+1}\right\} = 0. \tag{TVC}$$

But in this problem (NPG$'$) is the relevant restriction.

2. - Iterate on (FBC):

$$
\begin{aligned}
A_0 &= \frac{1}{1+r} A_1 - (y_0 - c_0) \\
&= \frac{1}{1+r} \left(\frac{1}{1+r} A_2 - (y_1 - c_1) \right) - (y_0 - c_0) \\
&= -\sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t (y_t - c_t) \\
&\quad + \left(\frac{1}{1+r} \right)^{T+1} A_{T+1},
\end{aligned}
$$

- Impose (NPG′):

$$
A_0 = -\sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t (y_t - c_t) \, .
$$

2. - Rearrange to derive the present-value budget constraint:

$$\sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^{t} c_t\left(I_t\right) = A_0 + \sum_{t=0}^{T} \left(\frac{1}{1+r}\right)^{t} y_t\left(I_t\right), \tag{PVBC}$$

which holds for all realized $\{y_t\}$.

- The right-hand-side of (PVBC) is often known as lifetime wealth.
- (PVBC) does not imply that the time path of $\{c_t\left(I_t\right)\}_{t=0}^{T}$ is known in advance.

3. Infinite-horizon consumer's problem

- The consumer solves

$$\max_{\{c_t\}_{t=0}^{\infty}} E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right)$$

subject to (FBC), (A0) and (NNG). Add either

$$\lim_{J \to \infty} E_t \left\{ \left(\frac{1}{1+r} \right)^J A_{t+J} \right\} = 0, \ \forall t, \qquad \text{(ENPG)}$$

or

$$\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j c_{t+j} (I_{t+j}) \qquad \text{(PVBC}'\text{)}$$

$$= A_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j y_{t+j} (I_{t+j}), \ \forall t.$$

3. Infinite-horizon consumer's problem
   - Note that we are restricting $\{y_t\}$ as well as $\{c_t\}$.
   - Aiyagari (1994) shows that (PVBC′) and (NNG) is equivalent to
     $$A_t \geq A_{\min}, \ \forall t. \qquad \text{(LC)}$$
   - Defer use of (LC). Impose (ENPG) instead.
   - Assume that y_t is a Markov process, so that
     $$F\left(y_t | \, y_{t-1}, y_{t-2}, \ldots\right) = F\left(y_t | \, y_{t-1}\right),$$
     and moreover, that $F\left(y_t | \, y_{t-1}\right)$ does not depend on *t*.
   - Suppose consumers choose c_t on the basis of
     $$I_t = \left\{ \{A_{t-j}, y_{t-j}\}_{j=0}^{t}, \{c_{t-j}\}_{j=1}^{t} \right\}.$$

3. - Bellman's equation is:

$$V(A_t, y_t) = \max_{c_t \geq 0} u(c_t) + \beta E\left\{ V(A_{t+1}, y_{t+1}) \middle| y_t \right\},$$

$$s.t. \quad A_{t+1} = (1+r)(A_t + y_t - c_t). \qquad \text{(FBC)}$$

- With $c_t > 0$, we get

$$\begin{aligned} u'(c_t) &= \beta(1+r) E_t\left(\frac{\partial V[t+1]}{\partial A_{t+1}}\right), \\ \frac{\partial V[t]}{\partial A_t} &= \beta(1+r) E_t\left(\frac{\partial V[t+1]}{\partial A_{t+1}}\right), \end{aligned}$$

(using Benveniste and Schienkman's result), so that

$$u'(c_t) = \beta(1+r) E_t\left(u'(c_{t+1})\right). \qquad \text{(EE)}$$

3. - Four key equations:

$$u'(c_t) = \beta(1+r)E_t\left(u'(c_{t+1})\right), \qquad \text{(EE)}$$

$$A_{t+1} = (1+r)(A_t + y_t - c_t), \qquad \text{(FBC)}$$

$$A_0 \text{ given}, \qquad \text{(A0)}$$

$$\lim_{J\to\infty} E_t\left\{\left(\frac{1}{1+r}\right)^J A_{t+J}\right\} = 0. \qquad \text{(ENPG)}$$

- (EE) and (FBC) give us the joint law of motion for (c_t, A_{t+1}).
- (A0) and (ENPG) provide two boundary conditions.

4. Quadratic utility and the random walk hypothesis

(a) Basic Results

- Suppose that

$$u(c_t) = ac_t - \frac{b}{2}c_t^2, \quad a, b > 0.$$

and assume that lifetime wealth is such that

$$0 < c_t < a/b \text{ ("bliss point")}, \ \forall t.$$

- Then(EE) becomes

$$\begin{aligned} a - bc_t &= \beta(1+r) E_t(a - bc_{t+1}), \\ E_t(c_{t+1}) &= \frac{a}{b}\left[1 - \frac{1}{\beta(1+r)}\right] + \frac{1}{\beta(1+r)} c_t. \end{aligned}$$

4. (a) Basic Results

- Under quadratic utility, (EE) becomes

$$E_t\left(c_{t+1}\right) = \frac{a}{b}\left[1 - \frac{1}{\beta\left(1+r\right)}\right] + \frac{1}{\beta\left(1+r\right)} c_t .$$

- Then

$$\begin{aligned} \beta\left(1+r\right) = 1 \Rightarrow & \\ c_t = E_t\left(c_{t+1}\right). & \qquad \text{(EE}'\text{)} \end{aligned}$$

- $\{c_t\}$ is a martingale process, and:

$$\begin{aligned} c_{t+1} &= c_t + \eta_{t+1}, \\ \eta_{t+1} &\equiv c_{t+1} - E_t\left(c_{t+1}\right), \\ E_t\left(\eta_{t+1}\right) &= 0. \end{aligned}$$

4. (a) Basic Results

- Under quadratic utility:

$$\begin{aligned} c_{t+1} &= c_t + \eta_{t+1}, \\ E_t\left(\eta_{t+1}\right) &= 0. \end{aligned}$$

- $\{\eta_{t+1}\}$ forms a martingale difference sequence.
- This is very similar—but not identical—to the random walk:

$$x_{t+1} = x_t + \varepsilon_{t+1},$$

where ε_{t+1} is white noise.

4. Quadratic utility

(b) Certainty-equivalence

- Optimal c_t under uncertainty $= E(c_t$ under certainty$)$.

$$u'(c_t) = \beta(1+r)E_t\left(u'(c_{t+1})\right) = \beta(1+r)u'\left(E_t(c_{t+1})\right).$$

- Reason: Under quadratic utility, $u'(c_{t+1})$ is linear in c_{t+1}.
- This is not true in general.

4. (c) Permanent Income

- Basic Result
  - In the finite-horizon case, with $J \equiv T - t$, (PVBC) implies

$$E_t \left\{ \sum_{j=0}^{J} \left(\frac{1}{1+r} \right)^j c_{t+j} \right\} = W_t. \quad \text{(EPVBC)}$$

$$W_t \equiv A_t + E_t \left\{ \sum_{j=0}^{J} \left(\frac{1}{1+r} \right)^j y_{t+j} \right\}.$$

  - Informal HW: show that as $J \to \infty$, (EPVBC) follows from (FBC) and (ENPG).

4. (c) Permanent Income

- Basic Result
  - Equation (EE′) and the law of iterated expectations imply that

$$\begin{aligned} E_t\left(c_{t+2}\right) &= E_t\left(E_{t+1}\left(c_{t+2}\right)\right) \\ &= E_t\left(c_{t+1}\right) \\ &= c_t, \end{aligned}$$

so that

$$\begin{aligned} E_t\left\{\sum_{j=0}^{J}\left(\frac{1}{1+r}\right)^j c_{t+j}\right\} &= c_t\sum_{j=0}^{J}\left(\frac{1}{1+r}\right)^j \\ &\equiv R_J c_t. \end{aligned}$$

4. (c) Permanent Income

- - (EPVBC) becomes

$$c_t = \frac{1}{R_J} W_t \equiv y_t^P.$$

where

$$\sum_{j=0}^{J} \left(\frac{1}{1+r} \right)^j y_t^P = W_t.$$

- Consumption equals permanent income.

4. (c) Permanent Income

- Implications
  - A temporary change in income leads to a permanent change in expected consumption: effects of income change are extended over time by consumption smoothing.
  - The effect of a change in current income on current consumption depends on how it affects permanent income: permanent changes have bigger effects.

4. (c) Permanent Income

- Friedman (1957)
  - Consider the linear projection of consumption on total income:
    $$\widehat{c}_t = \alpha + b y_t,$$
  - For a cross-section of households at a point in time, $\alpha > 0$, b is much less than 1.
  - For a country over time, $\alpha \approx 0$, b is closer to 1.
  - Define transitory income
    $$y_t^T = y_t - y_t^P.$$
  - Suppose $C\left(y_t^T, y_t^P\right) = 0$.

4. (c) Friedman (1957)

- We are considering: $c_t = \alpha + by_t$.
- The coefficient b follows

$$b = \frac{C\left(y_t, c_t\right)}{V\left(y_t\right)} = \frac{C\left(y_t^T + y_t^P, y_t^P\right)}{V\left(y_t^T + y_t^P\right)}$$

$$= \frac{V\left(y_t^P\right)}{V\left(y_t^P\right) + V\left(y_t^T\right)}.$$

- Cross-section data: $V\left(y_t^T + y_t^P\right)$ big relative to $V\left(y_t^P\right)$ because of wide variance of household transitory income.
- Time series data: $V\left(y_t^T + y_t^P\right)$ close to $V\left(y_t^P\right)$ because transitory income averages out across households in the aggregate.

4. (d) Hall's (1978) test

- Consider

$$\widehat{c}_{t+1} = \alpha + \beta c_t + \gamma x_t,$$

where x_t is some other variable known at time t.

- Recall that that under LQ preferences

$$E_t\left(c_{t+1}\right) = \frac{a}{b}\left[1 - \frac{1}{\beta\left(1+r\right)}\right] + \frac{1}{\beta\left(1+r\right)}c_t,$$

so that $\gamma = 0$.

- Some evidence that $\gamma \neq 0$.
- But suppose that returns are stochastic, so that

$$a - bc_t = \beta E_t\left\{\left(a - bc_{t+1}\right)\left(1 + r_{t+1}\right)\right\}.$$

Then γ probably $\neq 0$.

5. Precautionary Saving

(a) Coefficient of relative risk aversion:

- Measures curvature of utility function; elasticity of MU:

$$\gamma \equiv -\frac{\partial MU/MU}{\partial c/c} = -\frac{u''(c)\,c}{u'(c)},$$

- Constant relative risk aversion (CRRA) function:

$$u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \quad \lim_{\gamma\to 1}\frac{c^{1-\gamma}-1}{1-\gamma} = \ln(c).$$

5. (b) Precautionary saving

- Saving to offset future uncertainty.
- Rewrite the Euler equation

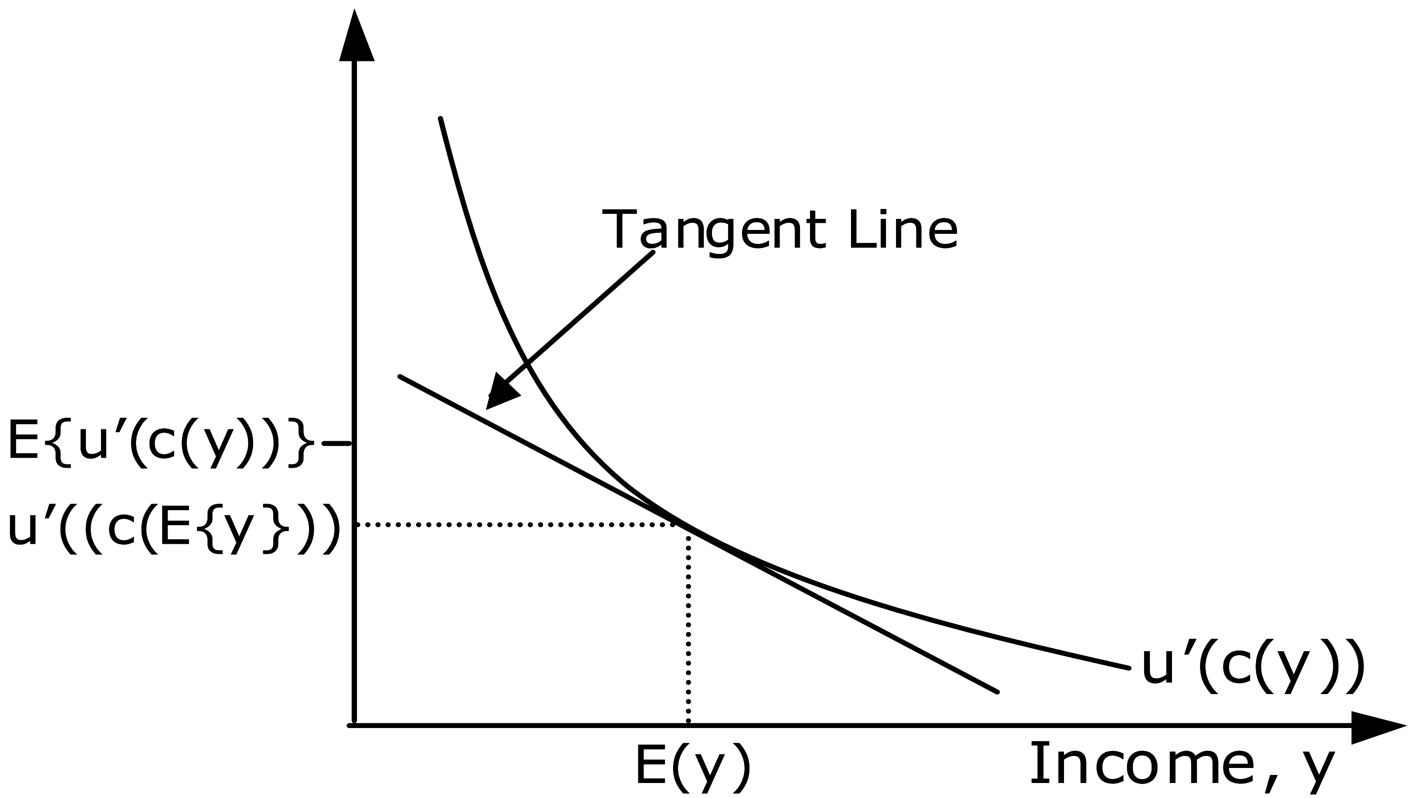
$$u'(c(\mathbf{x}_t)) = \beta E_t \left\{ u'(c(\mathbf{x}_{t+1}))(1 + r(\mathbf{x}_{t+1})) \right\},$$

  where $\mathbf{x}_{t+1}$ is the underlying random state vector (might include A_{t+1} and y_{t+1}).
- If $u'(c_{t+1})(1 + r_{t+1})$ is <u>convex</u> in $\mathbf{x}_{t+1}$, then by <u>Jensen's Inequality</u>:

$$\begin{aligned} u'(c_t) &= \beta E_t \left\{ u'(c_{t+1})(1 + r_{t+1}) \right\} \\ &> \beta u'(E_t(c_{t+1})) E_t(1 + r_{t+1}). \end{aligned}$$

- When $u'(c_{t+1})(1 + r_{t+1})$ is uncertain, $u'(c_t)$ is higher $\Rightarrow c_t$ is lower: more saving.

5. (b) - Graph for $r_{t+1} = r$.



- Comment: Jensen's inequality holds for vector functions as well.

5. (c) Key factor: Convex marginal utility, i.e., $u''' (c) > 0$.

- Measure curvature of marginal utility with Kimball's (1990) "coefficient of relative prudence":

$$-\frac{\partial u'' (c) / u'' (c)}{\partial c / c} = -\frac{u''' (c) \, c}{u'' (c)}.$$

- For a CRRA function with parameter γ, Kimball's measure is $1 + \gamma$.

(d) Intuition: $u''' (c) > 0 \Rightarrow$ gain in $u' (c)$ when future income is unexpectedly low more than offsets loss in $u' (c)$ when future income is unexpectedly high $\Rightarrow$ higher return to saving.

(e) Precautionary saving is <u>not</u> the same as risk-aversion.

(f) Implication: Higher uncertainty about future income increases saving.

6. Liquidity Constraints

(a) Theoretical justification

- Aiyagari (1994) shows that (PVBC) and (NNG) hold under uncertainty only if there is a borrowing limit.
- Also consistent with “casual empiricism”.

(b) Basic Model

- The borrowing constraint is

$$\frac{A_{t+1}}{1+r} = A_t + y_t - c_t \geq A_{\min},\ \forall t. \tag{LC}$$

- Assume:

$$\begin{aligned} -\infty &< y_{\min} \leq y_t,\ \forall t, \\ A_{\min} &\geq -\frac{y_{\min}}{r}. \end{aligned}$$

6. Liquidity Constraints

(b) Basic Model

- Define cash-on-hand, x_t, as

$$x_t = A_t + y_t - A_{\min}.$$

- Then we have:

$$V(x_t, y_t) = \min_{\mu_t \geq 0} \max_{c_t \geq 0} \; u(c_t) + \beta E\{V(x_{t+1}, y_{t+1}) | \, y_t\} + \mu_t (x_t - c_t),$$

$$s.t. \quad x_{t+1} = (1+r)(x_t - c_t) + y_{t+1} + rA_{\min}. \tag{FBC$'$}$$

6. (b) Basic Model

- Note that $x_{t+1} \geq 0$.
- With $c_t > 0$, we get

$$u'(c_t) = \beta(1+r) E_t\left(\frac{\partial V[t+1]}{\partial x_{t+1}}\right) + \mu_t$$

$$= \frac{\partial V[t]}{\partial x_t},$$

$$u'(c_t) = \beta(1+r) E_t\big(u'(c_{t+1})\big) + \mu_t, \qquad \text{(EE}''\text{)}$$

$$\mu_t \geq 0, \quad x_t - c_t \geq 0, \quad \mu_t(x_t - c_t) = 0. \qquad \text{(KTC)}$$

6. (c) Interpretation

- $\mu > 0 \Rightarrow$ that current MU > future MU $\Rightarrow$ current consumption too low because of borrowing limit.
- Iterate on (EE$''$) to get

$$u'(c_t) = \beta(1+r)E_t\left(\beta(1+r)u'(c_{t+2}) + \mu_{t+1}\right) + \mu_t.$$

- Even if the constraint does not bind today ($\mu_t = 0$), the possibility that it will bind tomorrow ($\mu_{t+1} > 0$) raises the effective return to saving $\Rightarrow$ lower consumption today (Zeldes, 1989).

6. Liquidity Constraints

(d) Deaton (1991)

- Notation and assumptions
  - Assume

$$b \equiv \beta(1+r) < 1,$$

so that liquidity constraints ultimately bind. If $b = 1$, agents save until constraints irrelevant.

  - Use (EE$''$) to define the function

$$\begin{aligned} w(x_t, y_t) &= u'(c_t) \\ &= \max\left\{u'(x_t), bE_t\left(u'(c_{t+1})\right)\right\}. \end{aligned}$$

6. (d) Deaton (1991)

- Stationary income
  - Assume:

$$y_t = \phi y_{t-1} + \varepsilon_t, \quad |\phi| < 1,$$

  - Modify (EE$''$) to get

$$w\left(x_t, y_t\right) = \max\left\{u'\left(x_t\right), bE_t\left(w\left(x_{t+1}, y_{t+1}\right)\right)\right\}.$$

  - Result 1: there exists a function $x^*\left(y\right)$ such that

$$\begin{aligned} c &= x, \quad x \leq x^*\left(y\right), \\ c &< x, \quad x > x^*\left(y\right). \end{aligned}$$

6. (d) Deaton (1991)

- Stationary income
  - Result 2: Saving is procyclical. During a boom $y_t > y_t^P$.
  - Result 3: as $\phi \to 1$, the degree of consumption smoothing decreases. (Reason: as swings become more persistent, cost of smoothing in terms of deferred consumption becomes more costly.)

6. (d) Deaton (1991)

- Non-stationary income
  - Income follows a random walk:

$$y_t = y_{t-1} + \varepsilon_t,$$

  - Optimal behavior is effectively to consume all current income (as $y_t = y_t^P$).
  - Once agents hit the borrowing constraint, they stay there forever.

6. (d) Deaton (1991)

- Correlated growth
  - Income growth follows

$$\Delta y_t = \phi \Delta y_{t-1} + \varepsilon_t, \quad \phi \in (0,1)$$

  - Saving is countercyclical. Saving increases at the beginning of a slump (as $y_t > y_t^P$), drops at the beginning of a boom (as $y_t < y_t^P$), and is zero during a long boom (constraint binds).

7. Life Cycle Model and Buffer Stock Saving

   (a) Life Cycle Model (Modigliani and Brunberg, 1954): pattern of saving over one's lifetime

      - Income pattern is hump-shaped: highest in middle ages.
      - Simple consumption smoothing model implies saving should be highest in middle age.
      - Under perfect foresight, consumption should not track income over the life cycle.

   (b) Data does not support this.

      - Consumption roughly tracks income: most households have only a small <u>buffer stock</u> of wealth.
      - People consume too much in middle age, too little in old age.

7. Life Cycle Model and Buffer Stock Saving

   (c) Explanations

- Rules of thumb
- Life cycle demographics
- Work-related expenses and consumption-leisure substitution.
- Government programs are implicit wealth (saving via Social Security)
- Uncertainty and a high discount rate.
    - High discount rate (relative to interest rates) works against saving.
    - Precautionary saving or liquidity constraints work against borrowing.