

Economics 701: Macroeconomics II
Spring 2009

Lecture 4: Asset Pricing

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1. Rational Expectations (Competitive) Equilibrium

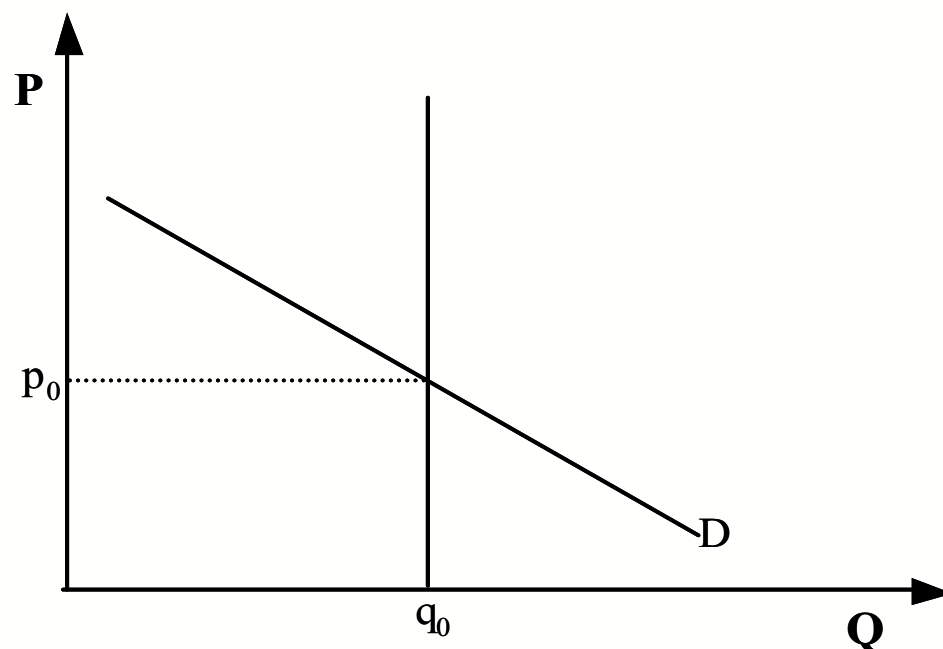
Given the set of exogenous stochastic process $\{x_t\}$, and initial conditions a_0 , a rational expectations equilibrium is a set of stochastic processes for prices $\{p_t\}$ and quantities $\{q_t\}$ such that:

- (a) Given $\{p_t\}$, $\{q_t\}$ is consistent with optimal behavior on the part of consumers, producers and (if relevant) the government.
- (b) Given $\{p_t\}$, $\{q_t\}$ satisfies the government's budget constraints and borrowing restrictions.
- (c) $\{q_t\}$ satisfies any market-clearing conditions.

2. Lucas “Tree” Model—Introduction

(a) Basic idea

- We are given equilibrium quantities and equilibrium demand functions—back out equilibrium prices.



we know the function $D(p)$ and the quantity q_0 : now find p_0 .

2. Lucas “Tree” Model—Introduction (continued)

(a) Basic idea

- Compare with Literature on Consumption
 - Consumption: Take rates of return as given, solve for consumption.
 - Asset Pricing: Take consumption as given, solve for rates of return.

2. (b) Model Structure

- Preferences: n identical consumers, maximizing

$$E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right),$$

$$\beta \in (0, 1), \quad u'(\cdot) > 0, \quad u''(\cdot) \leq 0.$$

- Endowment: one durable “tree” per individual. Each period, the tree yields some “fruit” ($d_t \equiv$ dividends).
- Technology: Fruit cannot be stored. Dividends are exogenous and follow a time-invariant Markov process

$$\Pr(d_{t+1} \leq y | d_t = x) = F(y, x), \quad \forall t,$$

with density $f(y, x)$.

2. (c) Solution strategy

- Find the competitive equilibrium allocation via the social planner's problem. Assume equal weights on each person's utility (parallel to equal endowments).
- Calculate the FOC for individuals with the opportunity to buy/sell specific assets.
- Evaluate FOC at the competitive equilibrium allocation.

2. (d) Step 1: Social planner's problem

- Use a representative agent
- Social planner solves

$$\max_{\{c_t\}_{t=0}^{\infty}} E_0 \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \right)$$

s.t. $c_t \leq d_t.$

- Solution: $c_t = d_t, \forall t$ (*non-storable good!*).

2. (e) Step 2: Representative consumer's problem.

$$\max_{\{c_t, b_{t+1}, s_{t+1}\}_{t=0}^{\infty}} E \left(\sum_{t=0}^{\infty} \beta^t u(c_t) \mid I_0 \right)$$

$$s.t. \quad c_t + p_t s_{t+1} + R_t^{-1} b_{t+1} = x_t,$$

$$x_t = (p_t + d_t) s_t + b_t,$$

$$\lim_{J \rightarrow \infty} \beta^J E_t (u'(c_{t+J}) p_{t+J} s_{t+J+1}) = 0$$

$$\lim_{J \rightarrow \infty} \beta^J E_t (u'(c_{t+J}) b_{t+J+1}) = 0,$$

$$s_0, b_0 \text{ given,}$$

2. (e) Step 2: Representative consumer's problem (continued)

where

c_t = consumption,

p_t = price of a tree = price of stock,
taken as given by consumer,

s_{t+1} = number of trees/shares of stock,

R_t = return on one-period risk-free bond,

R_t^{-1} = price of a one-period, risk-free discount bond,

b_{t+1} = risk-free discount bonds.

2. (e) Step 2: Representative consumer's problem (continued)

- Consumer i picks c_t^i , b_{t+1}^i and s_{t+1}^i on the basis of

$$I_t^i = \left\{ \begin{array}{l} \{d_{t-m}, p_{t-m}, R_{t-m}\}_{m=0}^t, \\ \{s_{t+1-m}^j, b_{t+1-m}^j\}_{m=0}^{t+1}, \forall j \neq i, \\ \{c_{t-m}^j, x_{t-m}^j\}_{m=0}^t, \forall j \neq i, \\ \{s_{t-m}^i, b_{t-m}^i, x_{t-m}^i\}_{m=0}^t, \{c_{t-m}^i\}_{m=1}^t, \end{array} \right\},$$

- It turns out that d_t summarizes the state of the aggregate economy, with $p_t = p(d_t)$ and $R_t = R(d_t)$.
- Because d_t is a time-invariant Markov process, the consumer's problem is time-invariant.

2. (e) Step 2: Representative consumer's problem (continued)

Bellman's functional equation:

$$\begin{aligned} V(x_t, d_t) = & \\ & \min_{\lambda_t \geq 0} \max_{c_t \geq 0, s_{t+1}, b_{t+1}} u(c_t) + \lambda_t (x_t - c_t - p_t s_{t+1} - R_t^{-1} b_{t+1}) \\ & + \beta \int V((p(d_{t+1}) + d_{t+1}) s_{t+1} + b_{t+1}, d_{t+1}) dF(d_{t+1}, d_t). \end{aligned}$$

The FOC for an interior solution are:

$$\begin{aligned} u'(c_t) &= \lambda_t, \\ \lambda_t p_t &= \beta \int \frac{\partial V[t+1]}{\partial x_{t+1}} (p(d_{t+1}) + d_{t+1}) dF(d_{t+1}, d_t), \\ \lambda_t R_t^{-1} &= \beta \int \frac{\partial V[t+1]}{\partial x_{t+1}} dF(d_{t+1}, d_t). \end{aligned}$$

2. (e) Step 2: Representative consumer's problem (continued)

Note that (by Benveniste-Scheinkman)

$$\frac{\partial V [t]}{\partial x_t} = \lambda_t,$$

so that

$$p_t = \beta E_t \left(\frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right),$$
$$R_t^{-1} = \beta E_t \left(\frac{u'(c_{t+1})}{u'(c_t)} \right).$$

2. (f) Step 3: Equilibrium

● Intuition

- Agents allocate resources on the basis of their beliefs about future prices and consumption
- These decision rules determine processes for market clearing prices and quantities.
- In a rational expectations equilibrium, the actual processes must be consistent with the beliefs.

2. (f) Step 3: Equilibrium

- **Sequential definition:** Given the stochastic process $\{d_t\}_{t=0}^{\infty}$ and the initial endowments $s_0 = 1$ and $b_0 = 0$, a rational expectations equilibrium consists of the stochastic processes $\{c_t, s_{t+1}, b_{t+1}, p_t, R_t\}_{t=0}^{\infty}$ such that:
 - Given the process for prices $\{p_t, R_t\}$, $\{c_t, s_{t+1}, b_{t+1}\}$ solves the consumer's problem.
 - All markets clear: $c_t = d_t$, $s_{t+1} = 1$, and $b_{t+1} = 0$, $\forall t$.

2. (f) Step 3: Equilibrium

- **Recursive definition:** given the random variable d_0 , the conditional distribution $F(d_{t+1}, d_t)$, and the initial endowments $s_0 = 1$ and $b_0 = 0$, a recursive rational expectations equilibrium consists of pricing functions $p(d)$ and $R(d)$, a value function $V(x, d)$, and decision functions $c(x, d)$, $s(x, d)$, and $b(x, d)$ such that:
 - Given the pricing functions $p(d)$ and $R(d)$, the value and policy functions $V(x, d)$, $c(x, d)$, $s(x, d)$, and $b(x, d)$ solve the consumer's problem.
 - Markets clear: for $x = p(d) + d$, $c(x, d) = d$, $s(x, d) = 1$, and $b(x, d) = 0$.

2. (f) Step 3: Equilibrium

- Find $R(d_t)$: impose the equilibrium allocation, $c_t = d_t$, to get

$$\begin{aligned} R_t^{-1} &= \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} \right) \\ &= \beta \frac{1}{u'(d_t)} E_t (u'(d_{t+1})) . \end{aligned} \quad (\text{EE})$$

- Find $p(d_t)$: impose the equilibrium allocation, $c_t = d_t$, to get

$$p_t = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1}) \right) . \quad (\text{EE}')$$

4. Lucas “Tree” Model—Interpretation

(a) Bond prices

- Recall equation (EE):

$$R_t^{-1} = \beta \frac{1}{u'(d_t)} E_t (u'(d_{t+1})), \quad (\text{EE})$$

$$R_t = \frac{u'(d_t)}{\beta E_t (u'(d_{t+1}))}.$$

- The price of a discount bond increases (return falls) in β .
- The price increases (return falls) in expected future marginal utility, and decreases in current marginal utility: reflects consumption smoothing motive.

4. Lucas “Tree” Model—Interpretation

(b) Stock prices

- Recall equation (EE′):

$$p_t = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1}) \right). \quad (\text{EE}')$$

(c) Equity premium

- The expected rate of return on stocks is

$$E_t (R_t^s) = E_t \left(\frac{p_{t+1} + d_{t+1}}{p_t} \right).$$

4. (c) Equity premium

- Recall that

$$E_t (XY) = E_t (X) E_t (Y) + Cov_t (X, Y) .$$

- Rewrite (EE'):

$$\begin{aligned} 1 &= \beta E_t \left(\frac{u' (d_{t+1})}{u' (d_t)} \left(\frac{p_{t+1} + d_{t+1}}{p_t} \right) \right) \\ &= \beta E_t \left(\frac{u' (d_{t+1})}{u' (d_t)} R_t^s \right) \\ &= \beta E_t \left(\frac{u' (d_{t+1})}{u' (d_t)} \right) E_t (R_t^s) \\ &\quad + Cov_t \left(\beta \frac{u' (d_{t+1})}{u' (d_t)}, R_t^s \right) . \end{aligned}$$

4. (c) Equity premium

- Insert (EE) and rearrange:

$$1 = R_t^{-1} E_t (R_t^s) + Cov_t \left(\beta \frac{u' (d_{t+1})}{u' (d_t)}, R_t^s \right),$$

$$E_t (R_t^s) = R_t - R_t Cov_t \left(\beta \frac{u' (d_{t+1})}{u' (d_t)}, R_t^s \right),$$

$$= R_t - \frac{u' (d_t)}{\beta E_t (u' (d_{t+1}))} Cov_t \left(\beta \frac{u' (d_{t+1})}{u' (d_t)}, R_t^s \right),$$

$$= R_t - \frac{Cov_t (u' (d_{t+1}), R_t^s)}{E_t (u' (d_{t+1}))}.$$

- The expected return on stocks equals the return on the risk-free bond plus the risk-premium $-\frac{Cov_t(\cdot, \cdot)}{E_t(\cdot)}$.

4. (c) Equity premium

- If the covariance $Cov_t(\cdot, \cdot)$ is negative, which we normally expect, there is an equity premium.
- Interpretation
 - The most desirable assets yield well when marginal utility is high ($Cov_t(\cdot, \cdot) > 0$). Risk-aversion means that agents prefer assets that act like insurance.
 - Investors are willing to sacrifice return if $Cov_t(\cdot, \cdot) > 0$, and they will demand higher returns if $Cov_t(\cdot, \cdot) < 0$.
- Mehra and Prescott (1985): the observed equity premium is difficult to reconcile with this formula. No accepted resolution.

4. (d) Consumption capital-asset pricing model (CAPM)
- If utility is quadratic:

$$u'(d_{t+1}) = a - bd_{t+1}.$$

the expected return on stock is given by

$$E_t(R_t^s) = R_t + \frac{b}{E_t(a - bd_{t+1})} Cov_t(d_{t+1}, R_t^s).$$

- $Cov_t(d_{t+1}, R_t^s)$ is the consumption beta.
- The standard CAPM focuses on the market beta = $Cov_t(R_t^m, R_t^s)$, where R_t^m is the return on the market portfolio.

4. (e) Stock prices: fundamentals and bubbles

- Rearrange equation (EE'):

$$p_t = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1}) \right),$$

$$E_t \left((1 - \beta L^{-1}) u'(d_t) p_t \right) = \beta E_t \left(u'(d_{t+1}) d_{t+1} \right).$$

- Recalling section 3, with $x_t = u'(d_t) p_t$, $z_t = -E_t(u'(d_{t+1}) d_{t+1})$, and $\lambda = \beta^{-1}$, we get

$$u'(d_t) p_t = \beta E_t \left(\sum_{j=0}^{\infty} \beta^j u'(d_{t+1+j}) d_{t+1+j} \right) + b_t,$$

$$E_t(b_{t+1}) = \beta^{-1} b_t,$$

so that b_t is a bubble term.

4. (e) Stock prices: fundamentals and bubbles

- It follows from the transversality condition,

$$\lim_{J \rightarrow \infty} \beta^J E_t (u' (d_{t+J}) p_{t+J}) = 0, \quad (\text{TVC})$$

that $b_t = 0$, and

$$p_t = E_t \left(\sum_{j=1}^{\infty} \beta^j \frac{u' (d_{t+j})}{u' (d_t)} d_{t+j} \right).$$

- Interpretation: The price of a share of stock equals the expected marginal-utility-adjusted PDV of its dividends.
- This is the fundamental (equilibrium) expression for stock prices.

4. (e) ● Special case 1: Linear utility
- Linear utility implies that

$$u'(d_t) = a,$$

$$p_t = E_t \left(\sum_{j=1}^{\infty} \beta^j d_{t+j} \right).$$

- This is the usual “fundamentals” solution.

4. (e) ● Special case 1: Linear utility
- Equation (EE') becomes

$$p_t = \beta E_t (p_{t+1} + d_{t+1}).$$

Over short time periods (e.g., weeks), $\beta \approx 1$, and $d_{t+1} \approx 0$, so that

$$E_t (p_{t+1}) \approx p_t,$$

and stock prices follow a martingale. This is the “random walk” theory of stock prices.

4. (e) ● Special case 1: Linear utility

- Shiller (1981) Test: Compare expected and realized dividend streams.
- Define

$$\begin{aligned} p_t^* &= \sum_{j=1}^{\infty} \beta^j d_{t+j} \\ &= p_t + \sum_{j=1}^{\infty} \beta^j (d_{t+j} - E_t d_{t+j}) \\ &\equiv p_t + u_t, \end{aligned}$$

with $E(u_t | p_t) = 0 \Rightarrow C(u_t, p_t) = 0$.

4. (e) ● Special case 1: Linear utility

- Then

$$V(p_t^*) = V(p_t) + V(u_t) > V(p_t),$$

- Shiller finds $V(p_t) \gg V(p_t^*)$.
- Possible explanations include bubbles and misspecified utility.
- Emerging consensus: volatility of p_t driven in large part by volatility in

$E_t \left(\sum_{j=1}^{\infty} \beta^j u'(d_{t+j}) / u'(d_t) \right) = \text{risk-free rates of return.}$

4. (e) Stock prices: fundamentals and bubbles

- Special case 2: logarithmic utility

$$\begin{aligned} p_t &= E_t \left(\sum_{j=1}^{\infty} \beta^j \frac{u'(d_{t+j})}{u'(d_t)} d_{t+j} \right) \\ &= E_t \left(\sum_{j=1}^{\infty} \beta^j \frac{d_t}{d_{t+j}} d_{t+j} \right) \\ &= \frac{\beta}{1-\beta} d_t. \end{aligned}$$

5. Contingent Claims

(a) Background

- Suppose that the state of the economy $\{I_t\}$ is a time invariant Markov process, following:

$$\begin{aligned} F(I_1, I_0) &= \Pr(I_{t+1} \leq I_1 | I_t = I_0) \\ &= \int_{-\infty}^{I_1} f(I_{t+1}, I_0) dI_{t+1}, \end{aligned}$$

when $F()$ is continuous and

$$F(I_1, I_0) = \sum_{I_{t+1} \leq I_1} f(I_{t+1}, I_0),$$

when $F()$ is discrete.

5. Contingent Claims

(a) Background

- Consider an asset that pays 1 unit of consumption when $I_{t+1} \in \mathcal{A}$.
- We will price the collection of such assets by a “pricing kernel” $q(I_{t+1}, I_t)$ that generates:

$$p_t^A = \int_{I_{t+1} \in \mathcal{A}} q(I_{t+1}, I_t) dI_{t+1}.$$

5. Contingent Claims

(b) The consumer's problem

- Let I' denote a realization of I_{t+1} , and let $y(I')$ denote the amount of state I' claims that the consumer buys.
- The consumer's wealth follows

$$x_{t+1}(I') = (p_{t+1}(I') + d_{t+1}(I'))s_{t+1} + y(I'),$$

- The consumer's flow budget constraint is:

$$c_t + p_t s_{t+1} + \int y(I_{t+1}) q(I_{t+1}, I_t) dI_{t+1} = x_t.$$

- The consumer chooses scalars for c_t and s_{t+1} , but a function for $y(I_{t+1})$.

5. Contingent Claims

(b) The consumer's problem

$$\begin{aligned} V(x_t, I_t) = & \min_{\lambda_t \geq 0} \max_{c_t \geq 0, s_{t+1}, y(\cdot)} u(c_t) \\ & + \lambda_t \left(x_t - c_t - p_t s_{t+1} - \int y(I_{t+1}) q(I_{t+1}, I_t) dI_{t+1} \right) \\ & + \beta \int V((p_{t+1}(I_{t+1}) + d_{t+1}(I_{t+1})) s_{t+1} + y(I_{t+1}), I_{t+1}) \\ & \quad \times f(I_{t+1}, I_t) dI_{t+1}, \end{aligned}$$

5. (b) The consumer's problem

- In this case, we can pick $y(\cdot)$ as if we were picking a scalar for each value of I_{t+1} .
- The FOC for an interior solution are

$$u'(c_t) = \lambda_t,$$

$$\lambda_t p_t = \beta \int \frac{\partial V[t+1]}{\partial x_{t+1}} [p_{t+1}(I_{t+1}) + d_{t+1}(I_{t+1})] \\ \times f(I_{t+1}, I_t) dI_{t+1},$$

$$\lambda_t q(I', I_t) = \beta \frac{\partial V[t+1]}{\partial x_{t+1}} f(I', I_t), \quad \forall I',$$

5. (b) The consumer's problem

• Since $\frac{\partial V[t]}{\partial x_t} = \lambda_t$, we have

$$p_t = \beta E_t \left(\frac{u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right),$$

$$q(I_{t+1}, I_t) = \beta \frac{u'(c_{t+1}(I_{t+1}))}{u'(c_t)} f(I_{t+1}, I_t), \quad \forall I_{t+1}.$$

5. Contingent Claims

(c) Equilibrium

- Impose $c_t = d_t$, $s_{t+1} = 1$, and $y(I_{t+1}) = 0$, $\forall I_{t+1}$.
- One can see that $I_t = d_t$, so that

$$p_t = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1}) \right).$$

$$q(d_{t+1}, d_t) = \beta \frac{u'(d_{t+1})}{u'(d_t)} f(d_{t+1}, d_t). \quad (\text{EE}'')$$

5. (c) Equilibrium

- Using (EE''), we can price any arbitrary asset.
- If an asset pays $w(d)$ units of consumption goods when $d_{t+1} = d$, its price is

$$\begin{aligned} p_t^w &= \int w(d_{t+1}) q(d_{t+1}, d_t) dd_{t+1} \\ &= \int w(d_{t+1}) \beta \frac{u'(d_{t+1})}{u'(d_t)} f(d_{t+1}, d_t) dd_{t+1} \\ &= \beta E_t \left(w(d_{t+1}) \frac{u'(d_{t+1})}{u'(d_t)} \right). \end{aligned}$$

- If this were not the case, there would be arbitrage opportunities.

5. (c) Equilibrium

- One-period risk-free bonds: $w(d_{t+1}) = 1$, so that

$$R_t^{-1} = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} \right).$$

- Stocks: $w(d_{t+1}) = p_{t+1} + d_{t+1}$, so that

$$p_t = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1} + d_{t+1}) \right).$$

- This shows us one way to price multi-period assets:

$$p_t^w = \beta E_t \left(\frac{u'(d_{t+1})}{u'(d_t)} (p_{t+1}^w + d_{t+1}^w(d_{t+1})) \right).$$

5. (d) Modigliani-Miller Theorem

- Consider a one-period “firm” that has acquired the rights to sell the dividends of one tree at time $t + 1$.
- At time t , the firm sells two types of claims, corporate bonds (b_t) and equities (e_t).
- The payoff function for a bond is

$$w^b(d_{t+1}) = \begin{cases} R, & \text{if } Rb_t \leq d_{t+1} \\ d_{t+1}/b_t, & \text{if } Rb_t > d_{t+1} \end{cases},$$

which implies that

$$p_t^b = \int_{d_{t+1} \geq Rb_t} Rq(d_{t+1}, d_t) dd_{t+1} + \int_{d_{t+1} < Rb_t} \frac{d_{t+1}}{b_t} q(d_{t+1}, d_t) dd_{t+1}.$$

5. (d) Modigliani-Miller Theorem

- The payoff function for an equity is

$$w^e(d_{t+1}) = \begin{cases} (d_{t+1} - Rb_t) / e_t, & \text{if } Rb_t \leq d_{t+1} \\ 0, & \text{if } Rb_t > d_{t+1} \end{cases},$$

which implies that

$$p_t^e = \int_{d_{t+1} \geq Rb_t} \frac{d_{t+1} - Rb_t}{e_t} q(d_{t+1}, d_t) dd_{t+1}.$$

5. (d) Modigliani-Miller Theorem

- Value of the firm:

$$\begin{aligned} V_t^F &= b_t p_t^b + e_t p_t^e \\ &= b_t \left[\int_{d_{t+1} \geq Rb_t} Rq(d_{t+1}, d_t) dd_{t+1} \right. \\ &\quad \left. + \int_{d_{t+1} < Rb_t} \frac{d_{t+1}}{b_t} q(d_{t+1}, d_t) dd_{t+1} \right] \\ &\quad + e_t \left[\int_{d_{t+1} \geq Rb_t} \frac{d_{t+1} - Rb_t}{e_t} q(d_{t+1}, d_t) dd_{t+1} \right] \\ &= \int d_{t+1} q(d_{t+1}, d_t) dd_{t+1}. \end{aligned}$$

5. (d) Modigliani-Miller Theorem

- The value of the firm, $V_t^F = \int d_{t+1} q(d_{t+1}, d_t) dd_{t+1}$, is independent of the number of bonds and equities outstanding.
- Modigliani-Miller Theorem: the value of a firm does not depend on how it is financed.
- If the firm issues more or less corporate bonds, the prices of bonds and equity change accordingly.
- Note that this result holds only under very restrictive assumptions: complete markets, symmetric information, no taxes.