BG

AMAT 327 Spring 2014 **Test No. 3**

The total number of points is 20. There are 3 problems.

Question 1 (8 points). Let $\phi\colon G \to H$ be a group homomorphism. Prove that the kernel $\ker(\phi)$ is a normal subgroup of G. So you need to show that it is a subgroup and that it is normal.

<u>Subgroup</u>: – nonempty b/c for any homom. φ $\varphi(e) = e'$, where e is the identity element in G and e' is the identity element in H

– (closure under the operation) if $a, b \in \ker(\varphi)$ then $\varphi(ab) = \varphi(a)\varphi(b) = e' \cdot e' = e'$ So $ab \in \ker(\varphi)$

– (closure under taking inverses) if $a \in \ker(\varphi)$ then $\varphi(a^{-1}) = \varphi(a)^{-1} = (e')^{-1} = e'$ So $a^{-1} \in \ker(\varphi)$.

<u>Normal</u> subgroup: given any $g \in G$ we want to check that $g^{-1}Kg \subseteq K$, so suppose $k \in K$ then $\varphi(g^{-1}kg) = \varphi(g)^{-1} \cdot \varphi(k) \cdot \varphi(g) =$

$$\varphi(g)^{-1} e' \varphi(g) = \varphi(g)^{-1}\varphi(g) = e'$$

So, indeed, $g^{-1}kg \in K$.

Question 2 (8 points). Let $\phi\colon G \to H$ be group homomorphism. Show that:

(a) $\ker(\phi)$ is the trivial subgroup $\{e\}$ if and only if ϕ is a monomorphism;

(b) if ϕ is a surjection and N is a normal subgroup of G then $\phi(N)$ is a normal subgroup of H.

(a) suppose $\ker(\varphi) = \{e\}$ and suppose $\varphi(a) = \varphi(b)$ then $\varphi(a)\cdot\varphi(b)^{-1} = e'$ in H

Since φ is a homom., $\varphi(ab^{-1}) = e'$, so

$ab^{-1} \in \ker(\varphi) = \{e\}$, so

$ab^{-1} = e \Rightarrow a = b \Rightarrow \varphi$ is a <u>mono</u>morphism

Suppose φ is a monomorphism and suppose $\varphi(a) = e'$ (that is, $a \in \ker(\varphi)$) we always have $\varphi(e) = e'$, so $a = e$. This means $\ker(\varphi) = \{e\}$.

(b) suppose $n' \in \varphi(N)$ and $h \in H$, then we want to see $h^{-1}n'h \in \varphi(N)$.

φ is onto $\Rightarrow h \in \varphi(g)$ for some $g \in G$

Now $n' \in \varphi(N)$ so there is $n \in N$ with $\varphi(n) = n'$. We have $\varphi(g^{-1}ng) = \varphi(g)^{-1}\varphi(n)\varphi(g)$ $= h^{-1}n'h$. On the other hand, $g^{-1}ng \in N$ because $N \triangleleft G \Rightarrow h^{-1}n'h = \varphi(g^{-1}ng) \in \varphi(N)$.

Question 3 (4 points). Suppose G is a group, and suppose K and L are two normal subgroups of G with the property that $K < L$. Then we have two factor groups G/K and G/L. Do this:

(a) prove that every right coset of K is contained in a right coset of L;

(b) explain how (a) tells you that there is a well-defined function from G/K to G/L;

(c) check that this function is actually a homomorphism.

(a) $K \subseteq L \Rightarrow Ka \subseteq La$

(b) define $\varphi : G/K \to G/L$ by $\varphi(Ka) = La$.

To check φ is well-defined, suppose $Ka = Ka'$ then $a' \in Ka \Rightarrow a' \in La$

But $K \subseteq L \Rightarrow Ka' \subseteq La$.

So the definition of φ is independent of the representative of the coset $Ka = Ka'$.

(c) Need $\varphi(Ka)\varphi(Kb) = \varphi(KaKb)$
$= \varphi(Kab)$

which is equivalent to $La \cdot Lb = Lab$

But this is true because L is normal (in fact, this is precisely the operation in G/L when L is normal).