The total number of points is 20. There are 3 problems.

**Question 1** (8 points). Let \( \phi: G \rightarrow H \) be a group homomorphism. Prove that the kernel \( \ker(\phi) \) is a normal subgroup of \( G \). So you need to show that it is a subgroup and that it is normal.

**Subgroup:**
- Nonempty: \( \ker(\phi) \) for any homomorphism \( \phi \), \( \phi(e) = e' \), where \( e \) is the identity element in \( G \) and \( e' \) is the identity element in \( H \).
- Closure under the operation: if \( a, b \in \ker(\phi) \) then \( \phi(ab) = \phi(a)\phi(b) = e' \cdot e' = e' \).
- So \( ab \in \ker(\phi) \).

**Normal subgroup:**
- Given any \( g \in G \) we want to check that \( g'Kg \subseteq K \), so suppose \( keK \).

Then \( \phi(g'kg) = \phi(g)^{-1}\phi(k)\phi(g) = \phi(g)^{-1}e'\phi(g) = \phi(g)^{-1}\phi(g) = e' \).

So, indeed, \( g'kg \in K \).
Question 2 (8 points). Let $\phi : G \to H$ be group homomorphism. Show that:

(a) $\ker(\phi)$ is the trivial subgroup $\{e\}$ if and only if $\phi$ is a monomorphism;

(b) if $\phi$ is a surjection and $N$ is a normal subgroup of $G$ then $\phi(N)$ is a normal subgroup of $H$.

(a) Suppose $\ker(\phi) = \{e\}$ and suppose $\phi(a) = \phi(b)$ then $(\phi(ab))^-1 = e'$ in $H$.

Since $\phi$ is a homomorphism, $\phi(ab^-1) = e'$, so $ab^-1 \in \ker(\phi) = \{e\}$, so $a = b \Rightarrow \phi$ is a monomorphism.

Suppose $\phi$ is a monomorphism and suppose $\phi(a) = e'$ (that is, $a \in \ker(\phi)$)

we always have $\phi(e) = e'$, so $a = e$.

Thus means $\ker(\phi) = \{e\}$.

(b) Suppose $n' \in \phi(N)$ and $h \in H$, then we want to see $h^{-1}n'h \in \phi(N)$.

$\phi$ is onto $\Rightarrow h \in \phi(g)$ for some $g \in G$

Now $n' \in \phi(N)$ so there is $n \in N$ with $\phi(n) = n'$. We have $\phi(g^{-1}ng) = \phi(g)^{-1}\phi(n)\phi(g)$

$= h^{-1}n'h$. On the other hand, $g^{-1}ng \in N$ because $N < G$ $\Rightarrow h^{-1}n'h = \phi(g^{-1}ng) \in \phi(N)$. 


Question 3 (4 points). Suppose $G$ is a group, and suppose $K$ and $L$ are two normal subgroups of $G$ with the property that $K < L$. Then we have two factor groups $G/K$ and $G/L$. Do this:

(a) prove that every right coset of $K$ is contained in a right coset of $L$;
(b) explain how (a) tells you that there is a well-defined function from $G/K$ to $G/L$;
(c) check that this function is actually a homomorphism.

(a) $K \subseteq L \Rightarrow Ka \subseteq La$

(b) define $\varphi : G/K \rightarrow G/L$ by $\varphi(Ka) = La$.

To check $\varphi$ is well-defined, suppose $Ka = Ka'$ then $a' \in Ka \Rightarrow a' \in La$

But $K \subseteq L \Rightarrow Ka' = La$.

So the definition of $\varphi$ is independent of the representative of the coset $Ka = Ka'$.

(c) Need $\varphi(Ka) \varphi(Kb) = \varphi(Kab)$

$= \varphi(Kab)$

which is equivalent to $La \cdot Lb = Lab$

But this is true because $L$ is normal (in fact, this is precisely the operation in $G/L$ when $L$ is normal).