BG

AMAT 327 Spring 2014 **Test No. 2**

The total number of points is 20. There are 3 problems.

Question 1 (6 points). Let G be any group.
(a) Prove that if an element g in G satisfies $g \cdot g = g$ then $g = e$.
(b) Prove that if G has no proper subgroups then G must be a cyclic group.

(a) Every element $g \in G$ has an inverse g^{-1}. I will multiply both sides of the equation $g \cdot g = g$ (and on the <u>same</u> side of the expressions!) by g^{-1}: $g^{-1}(g \cdot g) = g^{-1} \cdot g$. By associativity, $(g^{-1}g) \cdot g = g^{-1} \cdot g$. Using $g^{-1} \cdot g = e$, $e \cdot g = e \Rightarrow g = e$.

(b) Suppose $g \neq e$ in G. (If G has no such element then $G = \{e\}$ and so is cyclic.) Consider the cyclic subgroup $\langle g \rangle < G$. It cannot be proper but it is also not $\{e\}$ because $g \in \langle g \rangle$ so $\langle g \rangle = G$.

Question 2 (6 points). Let $G = S_4$ and let H be the subgroup of G which permutes the first three of the four elements and keeps the fourth fixed.

(a) Find the order of H.

(b) Find the index of H in G.

(a) I need to count the elements of H.
Each element fixes the 4th element of the set $\{1,2,3,4\}$ and permutes $\{1,2,3\}$.
There are no other constraints.
So H has the same # of elements as S_3.
Now $|H| = |S_3| = 3! = 6$.

(b) $|G| = |S_4| = 4! = 24$
So the index $i_G(H) = \frac{|H|}{|G|} = \frac{24}{6} = 4$.

Question 3 (8 points). Let G be any group, and let K and L be arbitrary subgroups of G. Prove that the intersection $K \cap L$ is also a subgroup of G.

- $K \cap L$ is nonempty because $e \in K$ and $e \in L \Rightarrow e \in K \cap L$
- suppose $g \in K \cap L$ and $h \in K \cap L$. This means that $g \in K, h \in K \Rightarrow gh \in K$ because $K < G$ and, independently, $g \in L, h \in L \Rightarrow gh \in L$ because $L < G$. Therefore, $gh \in K \cap L$.
- Suppose $g \in K \cap L$ then $g \in K, g \in L$

$$\left.\begin{array}{l} K < G \Rightarrow g^{-1} \in K \\ L < G \Rightarrow g^{-1} \in L \end{array}\right\} \Rightarrow g^{-1} \in K \cap L.$$

I have shown $K \cap L$ is nonempty, closed under the operation, and under taking inverses $\Rightarrow$ it is a subgroup of G.