The total number of points is 20. There are 3 problems.

Question 1 (6 points). Let $G$ be any group.
(a) Prove that if an element $g$ in $G$ satisfies $g \cdot g = g$ then $g = e$.
(b) Prove that if $G$ has no proper subgroups then $G$ must be a cyclic group.

(a) Every element $g \in G$ has an inverse $g'$. I will multiply both sides of the equation $g \cdot g = g$ (and on the same side of the expressions!) by $g'$: $g' (g \cdot g) = g' \cdot g$. By associativity, $(g' g) g = g' \cdot g$. Using $g' \cdot g = e$, $e \cdot g = e \Rightarrow g = e$.

(b) Suppose $g \neq e$ in $G$. (If $G$ has no such element then $G = \{ e \}$ and so is cyclic.) Consider the cyclic subgroup $\langle g \rangle < G$. It cannot be proper but it is also not $\{ e \}$ because $g \in \langle g \rangle$ so $\langle g \rangle = G$. 
Question 2 (6 points). Let \( G = S_4 \) and let \( H \) be the subgroup of \( G \) which permutes the first three of the four elements and keeps the fourth fixed.

(a) Find the order of \( H \).
(b) Find the index of \( H \) in \( G \).

\( (a) \) I need to count the elements of \( H \).
Each element fixes the 4th element of the set \( \{1, 2, 3, 4\} \) and permutes \( \{1, 2, 3\} \).
There are no other constraints.
So \( H \) has the same # of elements as \( S_3 \).
Now \( |H| = |S_3| = 3! = 6 \).

\( (b) \) \( |G| = |S_4| = 4! = 24 \)
So the index \( [G : H] = \frac{|H|}{|G|} = \frac{6}{24} = 4 \).
Question 3 (8 points). Let $G$ be any group, and let $K$ and $L$ be arbitrary subgroups of $G$. Prove that the intersection $K \cap L$ is also a subgroup of $G$.

- $K \cap L$ is nonempty because $e \in K$ and $e \in L \Rightarrow e \in K \cap L$

- Suppose $g \in K \cap L$ and $h \in K \cap L$.
  This means that $g \in K$, $h \in K \Rightarrow gh \in K$ because $K \leq G$ and, independently, $g \in L$, $h \in L \Rightarrow gh \in L$ because $L \leq G$. Therefore, $gh \in K \cap L$.

- Suppose $g \in K \cap L$ then $g \in K$, $g \in L$
  $K \leq G \Rightarrow g' \in K \Rightarrow g' \in K \cap L$
  $L \leq G \Rightarrow g' \in L \Rightarrow g' \in K \cap L$

I have shown $K \cap L$ is nonempty, closed under the operation, and under taking inverses $\Rightarrow$ it is a subgroup of $G$. 