This is a solution of the problem I didn't finish in class.

The problem was to show that $n^{101} - n$ is always divisible by 33.

What I could do was to show this when n is a unit modulo 33:

then $n^{\varphi(33)} = n^{20} \equiv 1 \bmod 33$, so

$n^{100} = (n^{20})^5 \equiv 1 \bmod 33$, and

$n^{101} \equiv n \bmod 33$.

This is not enough to show $33 \mid n^{101} - n$ for all n because $\mathbb{Z}/33\mathbb{Z}$ has zero divisors. Instead, I will use the fact that $33 = 3 \cdot 11 = [3, 11]$ and a familiar strategy using Fermat's

Theorem for the primes 3 and 11.

It is enough to check that $3 \mid n^{101} - n$ and $11 \mid n^{101} - n$ for all n, because these facts are equivalent to

$n^{101} \equiv n \mod 3$, $n^{101} \equiv n \mod 11$

So then $n^{101} \equiv n \mod [3, 11] = 33$.

So here we go:

- When n is a unit mod 3 then
  $n^2 \equiv 1 \mod 3 \Rightarrow n^{100} = (n^2)^{50} \equiv 1 \mod 3$
  So $n^{101} \equiv n \mod 3$
- When n is not a unit then $3 \mid n \Rightarrow$
  Still $n^{101} \equiv n \equiv 0 \mod 3$.
- if n is a unit mod 11 then
  $n^{10} \equiv 1 \mod 11 \Rightarrow n^{100} = (n^{10})^{10} \equiv 1 \mod 11$

So $n^{101} \equiv n \mod 11$

- if $11 \mid n$ then still $n^{101} \equiv n \equiv 0 \mod 11$.

This verifies all cases.