The total number of points is 25.

**Question 1** (8 points). -
(a) Classify all non-zero elements in \( \mathbb{Z}/15\mathbb{Z} \) into units and zero divisors.
(b) Pick any unit that is not \([1]\) or \([14]\) from your list and find its order.
(c) Find the inverse of the same element from part (b).
(d) Find the complementary zero divisors for \([5]\).

**Question 2** (5 points). Show that the order of \([2]\) in \( \mathbb{Z}/125\mathbb{Z} \) is at least 25.

**Question 3** (6 points). Show that \( n^{19} - n \) is divisible by 7 for all positive integers \( n \).

**Question 4** (6 points). -
(a) State Euler's Theorem.
(b) Compute the value of Euler's function for \( m = 100 \) and restate Euler's Theorem explicitly in the case of \( \mathbb{Z}/100\mathbb{Z} \).

**SOLUTIONS**

1. @15 = 3 5 so all zero divisors are represented by integers divisible by 3 or 5, so here
   1 2 3 4 5 6 7 8 9 10 11 12 13 14
   u u z u z z u u z u z u u
   (u for "unit", z for "zero divisor")

2. I pick \([7]\) then \( 7^2 = 49 \equiv 4 \mod 15 \)
   \( 7^3 = 4 \cdot 7 = 28 \equiv 13 \mod 15 \)
   \( 7^4 = 13 \cdot 7 = 91 \equiv 1 \mod 15 \)
   So \( o ([7]) = 4 \)

   from the computation in (b).
(6) $[5]x = [0]$ gives for $[x] = x$ the equation $5x + 15y = 0$ (for some $y$).

$x = 0 + \frac{15}{5, 15} k = 3k$, so $X$ is any of $[0], [3], [6], [9], [12]$, so the zero divisors compl. to $[5]$ are $[3], [6], [9], [12]$. 125 = $5^3$, so $\varphi(125) = 5^3 - 5^2 = 100 = 2^2 \cdot 5^2$.

By Euler, $2^{100} \equiv 1 \mod 125$, so $o([2]) | 100 \Rightarrow o([2]) = \text{either } 2, 4, 5, 10, 25, 50, \text{ or } 100$.

All $2^2, 2^4, 2^5 = 32$ are $< 125$ and $2^{10} = (2^5)^2 = 32 = 1,024 \equiv 24 \mod 125$.

So $o([2]) \geq 25$.

(3) If $7 | n$ then $7 | n^{19} - n$.

If $7 | n$ then $[n]$ is a unit in $\mathbb{Z}_7$. 
So $n^6 \equiv 1 \mod 7$ by Fermat.
So $n^{18} = (n^6)^3 \equiv 1 \mod 7$
So $n^{19} \equiv n \mod 7$.

(4) If $[a]$ is a unit in $\mathbb{Z}/m\mathbb{Z}$ then
$[a] \varphi(m) = [1]$ in $\mathbb{Z}/m\mathbb{Z}$.

(6) Since $m = 100 = 2^2 \cdot 5^2$ then
$\varphi(m) = \varphi(2^2) \cdot \varphi(5^2) = (2^2 - 2)(5^2 - 5^1) = 2 \cdot 20 = 40$.

So Euler's Theorem says, in this case, if $[a]$ is a unit in $\mathbb{Z}/100\mathbb{Z}$ then $[a]^{40} = [1]$. 