Walsh Chapter 4

Part 2: Optimal Taxation and Seigniorage
1 Partial Equilibrium Model

- Assumptions

  - Define seigniorage

    \[ s_t = \frac{M_t - M_{t-1}}{P_t} = m_t - \frac{m_{t-1}}{1 + \pi_t} \]

  - Budget constraint with exogenous fixed \( g \)

    \[ b_t = Rb_{t-1} + g - \tau_t - s_t \]
– IBC

\[ E_t \sum_{t=0}^{\infty} R^{-1} (\tau_{t+i} + s_{t+i}) = Rb_{t-1} + \frac{R}{R-1} g \]

– Ad hoc loss function due to distortionary inflation and ordinary taxes

\[ \frac{1}{2} E_t \sum_{t=0}^{\infty} R^{-1} \left( (\tau_{t+i} + \phi_{t+i})^2 + (s_{t+i} + \epsilon_{t+i})^2 \right) \]

where marginal costs of both types of taxes are assumed to vary stochastically.
• First order conditions, letting $\lambda$ be multiplier on IBC

$$E_t (\tau_{t+i} + \phi_{t+i}) = \lambda$$

$$E_t (s_{t+i} + \epsilon_{t+i}) = \lambda$$

• Equalize marginal distortionary taxes across
  
  – instruments $\tau$ and $s$

  – time
• Tax smoothing over time for both instruments (Barro)

  – inflation and taxes should move together when revenue needs change

    * positively correlated for US and Japan, but negatively for France, Germany, UK (Poterba and Rotemberg 1990)

  – if $E_t \epsilon_{t+1} = \epsilon_t$, then seigniorage is a random walk

    * Mankiw (1987) finds inflation is a random walk in US

  – temporary expenditure seems to be financed with seigniorage (wars), whereas optimality would finance it with debt (smoothing principle says that should adjust to only permanent expenditure changes)

• Inflation surprises are not distortionary – if wars are unexpected $g$, perhaps optimal to finance them by letting $P$ jump up
2 Ramsey Problem with MIU

- Consumer’s problem
  
  – choose $c$, $l$, $m$ to maximize

  $$u(c, l, m)$$

  subject to

  $$f(n) \geq (1 + \tau) c + \tau mm$$

  where

  $$n = 1 - l \quad \tau_m = \frac{i}{1 + i}$$
- Lagrangian

\[ L = u(c, l, m) - \lambda [(1 + \tau) c + \tau m m - f(n)] \]

- First order conditions

\[ u_c = \lambda (1 + \tau) \]
\[ u_m = \lambda \tau m \]
\[ u_l = \lambda f'(n) \]
• Government’s problem

  – choose $\tau$, $\tau_m$ to maximize agent’s utility

    * subject to government budget constraint

      $$\tau c + \tau_m m \geq g$$

    * and economy’s resource constraint

      $$f \left(1 - l\right) \geq c + g$$
• Dual approach treats tax rates as controls and uses indirect utility

• Lagrangian using indirect utility function

\[
\max_{\tau, \tau_m} \left[ v(\tau, \tau_m) + \mu [\tau c(\tau, \tau_m) + \tau_m m(\tau, \tau_m) - g] + \theta [f(1 - l(\tau, \tau_m)) - c(\tau, \tau_m) - g] \right]
\]

where

\[
v(\tau, \tau_m) = u[c(\tau, \tau_m), m(\tau, \tau_m), l(\tau, \tau_m)]
\]

and all household choice variables have been expressed as a function of the tax variables
First order conditions

\[ \tau \quad v_{\tau} + \mu [c + \tau c_{\tau} + \tau m m_{\tau}] - \theta \left[ f' l_{\tau} + c_{\tau} \right] \leq 0 \]

\[ \tau_m \quad v_{\tau m} + \mu [m + \tau c_{\tau m} + \tau m m_{\tau m}] - \theta \left[ f' l_{\tau m} + c_{\tau m} \right] \leq 0 \]

where

\[ v_{\tau} = u_{c} c_{\tau} + u_{m} m_{\tau} + u_{l} l_{\tau} \]

\[ v_{\tau m} = u_{c} c_{\tau m} + u_{m} m_{\tau m} + u_{l} l_{\tau m} \]

- If FO condition on \( \tau_m \) is negative with \( \tau_m = 0 \), optimal not to tax money

- Resource constraint with equality implies last term in \( [] \) is zero

\[ u_{c} c_{\tau} + u_{m} m_{\tau} + u_{l} l_{\tau} + \mu [c + \tau c_{\tau} + \tau m m_{\tau}] \leq 0 \]
\[ u_c c_{\tau m} + u_m m_{\tau m} + u_l l_{\tau m} + \mu [m + \tau c_{\tau m} + \tau m m_{\tau m}] \leq 0 \]

- Write

\[ v_{\tau} = u_c c_{\tau} + u_m m_{\tau} + u_l l_{\tau} = u_l \left( \frac{u_c}{u_l} c_{\tau} + \frac{u_m}{u_l} m_{\tau} + l_{\tau} \right) , \]

where from agent's FO conditions

\[ \frac{u_c}{u_l} = \frac{1 + \tau}{f'(n)}, \quad \frac{u_m}{u_l} = \frac{\tau m}{f'(n)} \]

- Substitute

\[ v_{\tau} = u_l \left( \frac{1 + \tau}{f'(n)} c_{\tau} + \frac{\tau m}{f'(n)} m_{\tau} + l_{\tau} \right) = \frac{u_l}{f'(n)} \left( (1 + \tau) c_{\tau} + \tau m m_{\tau} + f'(n) l_{\tau} \right) \]
• Differentiate the agent’s budget constraint with respect to $\tau$ and set the result to zero. After taxes adjust, $c, m,$ and $l$ must adjust to keep the budget constraint satisfied.

\[
c + (1 + \tau) c_\tau + \tau m m_\tau + f' l_\tau = 0
\]

• Substitute into $v_\tau$ and then use the FO condition on $\tau$ set to zero

\[
v_\tau = -\frac{w l c}{f'(n)} = -\mu [c + \tau c_\tau + \tau m m_\tau]
\]

• Similar steps yield

\[
v_{\tau m} = -\frac{w l m}{f'(n)} = -\mu [m + \tau c_{\tau m} + \tau m m_{\tau m}]
\]
• Interior solutions imply

\[
\frac{v_{\tau m}}{v_{\tau}} = \left( \frac{m}{c} \right) = \left( \frac{m + \tau c_{\tau m} + \tau m m_{\tau m}}{c + \tau c_{\tau} + \tau m m_{\tau}} \right)
\]

- taxes should equalize the marginal rates of substitution in utility and transformation in providing government revenue

- \( \frac{v_{\tau m}}{v_{\tau}} = \left( \frac{m}{c} \right) \) is marginal rate of substitution (effects on utility) between two tax rates

- \( \left( \frac{m + \tau c_{\tau m} + \tau m m_{\tau m}}{c + \tau c_{\tau} + \tau m m_{\tau}} \right) \) is the marginal rate of transformation (from budget constraints)

• Equate marginal utility costs of raising revenue with each tax

\[
\frac{m}{m + \tau c_{\tau m} + \tau m m_{\tau m}} = \frac{c}{c + \tau c_{\tau} + \tau m m_{\tau}}
\]
Corner solutions

- If FO condition has inequality on money, then $\tau_m = 0$, and

$$\left(\frac{m}{c}\right) > \left(\frac{m + \tau c_{\tau m}}{c + \tau c_{\tau}}\right)$$

- Implies that optimal tax on money is zero if above inequality holds

- If utility is separable in money and consumption, then $c_{\tau m} = 0$. Since $c_{\tau} < 0$, the inequality fails, and money should be taxed

- If $c_{\tau m} > 0$, optimal to tax money

  * If $u_{cm} < 0$ (money and consumption are substitutes), then $c$ increases as $\tau_m$ increases and $c_{\tau m} > 0$
– If $c_{\tau m} < 0$, due to $u_{cm} > 0$ (money and consumption are comple-
ments), then optimality of taxing money depends on magnitudes
If introduce tax on labor supply, homothetic preferences in money and consumption and utility separable in leisure

- optimal tax on money is zero

- Friedman – ratio of marginal utility of money to consumption, given by $\frac{i}{1+i}$ should equal ratio of marginal production cost, given by zero
3 Ramsey Problem with CIA

• Consumer’s problem

  – $c_1$ is a cash good and $c_2$ is a credit good
  
  – choose $c_{1,t+i}$, $c_{2,t+i}$, $l_{t+i}$, $m_{t+i}$ to maximize

\[
E_t \sum_{i=0}^{\infty} \beta^i U \left( c_{1,t+i}, c_{2,t+i}, l_{t+i} \right)
\]
subject to budget constraint

\[
(1 + \tau_t^c) Q_t (c_{1t} + c_{2t}) + M_t + B_t
\]

\[
= \left(1 - \tau_t^h\right) Q_t (1 - l_t) + (1 + i_{t-1}) B_{t-1} + M_{t-1}
\]

* \(Q\) is producer price of output

* \(Y_t = 1 - l_t\), implying constant returns to scale in production

CIA constraint

\[
c_{1t} \leq \frac{m_{t-1}}{1 + \pi_t}
\]
Case 1: let $c_2 = 0$ – only cash goods

- the nominal interest rate is a tax on labor since receive wage and must hold it as money for one period before can buy good

- already have a tax on labor given by $\tau_t^h$, so do not need another

- optimal to set $i = 0$, and have no tax on money
• Case 2: let \( c_2 > 0 \) and assume preferences are homothetic

  – the nominal interest rate is a tax on \( c_1 \) relative to \( c_2 \)

  – at the optimum, marginal rate of substitution for the two goods should equal marginal rate of transformation, which is unity

  \[
  \frac{U_1}{U_2} = 1 + i
  \]

  – set \( i = 0 \)

  – How reasonable is homotheticity? depends on what is a cash good and what is a credit good
4 Ramsey Problem with Shopping Time Model

- If money is an intermediate input, and if final goods can be taxed, optimal not to tax intermediate inputs (distorts production efficiency)

- Assumptions
  - single unit of labor can be used for leisure, $l$, for shopping, $n^s$, or for production, $n$
    \[ l_t + n^s_t + n_t = 1 \]
  - shopping time production function
    \[ n^s_t = G(c_t, m_t) \]
* assume $G$ is homogeneous of degree $\eta$

$$G(\lambda tc_t, \lambda tm_t) = \lambda_t^\eta G(c_t, m_t)$$

* assume $\lambda_t = \frac{1}{c_t}$

$$n_t^s = G(c_t, m_t) = c_t^\eta G\left(1, \frac{m}{c}\right) = c_t^\eta g\left(\frac{m}{c}\right)$$

* assume $g$ is convex such that shopping time is non-increasing in money and money has a diminishing marginal product in reducing shopping time

$$g' \leq 0 \quad g'' \geq 0$$

* assume satiation level of money such that

$$g'\left(\frac{m}{c}\right) = 0 \text{ for } \frac{m}{c} \geq \bar{\mu}$$
– production function is CRS

\[ f(n) = 1 - l - n^s \]

• Agent’s problem

– choose \( c, n, d, m \) to maximize

\[
\sum_{i=0}^{\infty} \beta^i u \left[ c_{t+i}, 1 - n_{t+i} - c_{t+i}^\eta g \left( \frac{m_{t+i}}{c_{t+i}} \right) \right]
\]

– subject to flow budget constraint

\[
w_t = \left( \frac{1 + i_{t-1}}{1 + \pi_t} \right) d_{t-1} - \frac{i_{t-1}}{1 + \pi_{t-1}} m_{t-1} = c_t + d_t - (1 - \tau) f(n)
\]
- value function

\[ V(w_t) = \max \left\{ u \left[ c_t, 1 - n_t - c_t^n g \left( \frac{m_t}{c_t} \right) \right] + \beta V(w_{t+1}) \right\} \]

- maximize value function subject to

\[ w_t - [c_t + d_t - (1 - \tau) f(n_t)] = 0 \quad \text{with multiplier } \lambda_t \]

where

\[ w_{t+1} = \left( \frac{1 + i_t}{1 + \pi_{t+1}} \right) d_t - \frac{i_t}{1 + \pi_{t+1}} m_t \]
- FO conditions

\[ c_t \quad u_c - u_l \left( \eta c_t^{\eta-1} g + c_t^{\eta} g' \left( \frac{-m_t}{c_t^2} \right) \right) - \lambda_t = 0 \]

\[ \lambda_t = u_c - u_l c_t^{\eta-1} \left( \eta g - g' \frac{m_t}{c_t} \right) \]

\[ n_t \quad - u_l + (1 - \tau) \lambda_t = 0 \]

\[ (1 - \tau) \lambda_t = u_l \]

\[ d_t \quad - \lambda_t + \beta V_w (w_{t+1}) \frac{1 + i_t}{1 + \pi_{t+1}} = 0 \]
- envelope condition

\[ V_w (w_{t+1}) = \lambda_{t+1} \]

- define

\[ R_t = \frac{1 + i_t}{1 + \pi_{t+1}} \]

\[ \lambda_t = \beta R_t \lambda_{t+1} \]

\[ m_t = u \eta c^\eta g' - \beta \lambda_{t+1} \frac{i_t}{1 + \pi_{t+1}} = 0 \]

\[ \beta \lambda_{t+1} = \frac{\lambda_t}{R_t} \]

\[ \lambda_t \frac{i_t}{1 + i_t} = -u \eta c^\eta g' \]
• Set up government’s problem

  – Agent’s flow budget constraint solved for $d_t$, with limit term set equal to zero

    $d_t = (1 + r_{t-1})d_{t-1} - \frac{\lambda_t-1}{1 + \pi_t} m_{t-1} - c_t + (1 - \tau_t)(1 - l_t - n_t^s)$

  – Present value of agent budget constraint

    $$(1 + r_{t-1})d_{t-1}$$

    $$= \sum_{i=0}^{\infty} D_i \left\{ c_{t+i} - (1 - \tau_{t+i}) (1 - l_{t+i} - n_{t+i}^s) + R_{t+i-1}I_{t+i-1}m_{t+i-1} \right\}$$

    where

    $$D_i = \prod_{j=1}^{i} R_{t+j-1}^{-1} \quad D_0 = 1 \quad I_t = \frac{i_t}{1 + i_t}$$

  – Assume $d_{t-1} = 0$; otherwise optimal to completely inflate debt away
Multiply and divide each term in IBC by $\lambda_{t+i}$

$$\sum_{i=0}^{\infty} \frac{D_i}{\lambda_{t+i}} \left[ c_{t+i} - (1 - \tau_{t+i})(1 - l_{t+i} - n_{i+i}^s) + R_{t+i}I_{t+i}m_{t+i-1} \right] \lambda_{t+i} = 0$$

Think about $D_i$

$$D_0 = 1; \quad D_1 = \frac{1}{R_1} \quad D_2 = \frac{1}{R_1R_2}$$

Use FO condition on debt solved for $\frac{1}{R_t}$

$$\frac{1}{R_t} = \beta \frac{\lambda_{t+1}}{\lambda_t} = D_1$$

$$D_2 = \frac{1}{R_1R_2} = \beta \frac{\lambda_{t+1}}{\lambda_t} \beta \frac{\lambda_{t+2}}{\lambda_{t+1}} = \beta^2 \frac{\lambda_{t+2}}{\lambda_t}$$
\[ D_i = \beta^i \frac{\lambda_{t+i}}{\lambda_t} = \beta^i D_0 \frac{\lambda_{t+i}}{\lambda_t} \quad \text{since} \quad D_0 = 1 \]

\[
\frac{D_i}{\lambda_{t+i}} = \frac{\beta^i}{\lambda_t}
\]

– can show that footnote 42, p. 187

\[
\sum_{i=0}^{\infty} D_i [R_{t+i-1} I_{t+i-1} m_{t+i-1}] = \sum_{i=0}^{\infty} \frac{\beta^i}{\lambda_t} [\lambda_{t+i} I_{t+i} m_{t+i}]
\]

– Substitute into the agent IBC

\[
0 = \sum_{i=0}^{\infty} \beta^i \left[ \lambda_{t+i} c_{t+i} - \lambda_{t+i} (1 - \tau_{t+i}) \left(1 - l_{t+i} - n^s_{t+i}\right) + \lambda_{t+i} I_{t+i} m_{t+i} \right]
\]
– Substitute agent’s FO conditions for $\lambda_{t+i}$

$$\sum_{i=0}^{\infty} \beta^i \left[ \left( u_c - u_l c_{t+i}^{\eta-1} \left( \eta g - g' \frac{m_{t+i}}{c_{t+i}} \right) \right) c_{t+i} 
\quad - u_l \left( 1 - l_{t+i} - n_{t+i}^s \right) - m_{t+i} u_l c_{t+i}^{\eta-1} g' \right]$$

$$= \sum_{i=0}^{\infty} \beta^i \left[ \left( u_c c_{t+i} - u_l c_{t+i}^{\eta} \eta g \right) - u_l \left( 1 - l_{t+i} - n_{t+i}^s \right) \right] = 0$$

– Recall

$$n_{t}^{s} = c_{t}^{\eta} g \left( \frac{m}{c} \right)$$

– Yielding the implementability condition

$$0 = \sum_{i=0}^{\infty} \beta^i \left[ u_c c_{t+i} - u_l \left( 1 - l_{t+i} \right) + u_l \left( 1 - \eta \right) c_{t+i}^{\eta} g \left( \frac{m_{t+i}}{c_{t+i}} \right) \right]$$
– Government Problem

* Choose $c_{t+i}$, $l_{t+i}$, and $m_{t+i}$ to maximize agent utility

$$\sum_{i=0}^{\infty} \beta^i u[c_{t+i}, l_{t+i}]$$

* subject to implementability condition

* and resource constraint

$$c_t + g_t \leq 1 - l_t - c_t^\eta g \left( \frac{m_t}{c_t} \right)$$

* FO condition on $m_{t+i}$

$$\left( \beta^i u_l (1 - \eta) \Psi - \mu_{t+i} \right) g' = 0$$
* Need \( g' = 0 \), that is, money must be held to satiation, which will require a zero nominal interest rate that is no tax on money.
5 Inflation and non-indexed tax systems

• Assumptions

  – Let nominal taxable income be given by

\[ Y_t = P_t f (k_{t-1}) + i_{t-1} B_{t-1} + P_t T_t + (P_t - P_{t-1}) (1 - \delta) k_{t-1}, \]

where last term denotes capital gains

  – Budget constraint becomes

\[
(1 - \tau) [P_t f (k_{t-1}) + i_{t-1} B_{t-1} + P_t T_t + (P_t - P_{t-1}) (1 - \delta) k_{t-1}] \\
= P_t c_t + P_t k_t - P_t (1 - \delta) k_{t-1} + (B_t - B_{t-1}) + (M_t - M_{t-1})
\]
Expressing in real terms

\[
(1 - \tau) \left[ f(k_{t-1}) + T_t + \frac{i_{t-1}b_{t-1}}{1 + \pi_t} + \frac{\pi_t}{1 + \pi_t} (1 - \delta) k_{t-1} \right]
\]

\[
= c_t + k_t - (1 - \delta) k_{t-1} + b_t - \frac{b_{t-1}}{1 + \pi_t} + m_t - \frac{m_{t-1}}{1 + \pi_t}
\]

\[
= (1 - \tau) \left[ f(k_{t-1}) + T_t + \frac{i_{t-1}b_{t-1}}{1 + \pi_t} + \frac{\pi_t}{1 + \pi_t} k_{t-1} \right]
\]

\[
+ (1 - \delta) k_{t-1} + \frac{b_{t-1}}{1 + \pi_t} + \frac{m_{t-1}}{1 + \pi_t}
\]

\[
= (1 - \tau) [ f(k_{t-1}) + T_t ] + \left(1 + \frac{\pi_t (1 - \tau)}{1 + \pi_t} \right) (1 - \delta) k_{t-1}
\]

\[
+ \left(1 + (1 - \tau) i_{t-1} \right) b_{t-1} + \frac{m_{t-1}}{1 + \pi_t}
\]
– In steady state FO condition on capital implies

\[(1 - \tau) f_k(k_{t-1}) + \left(1 + \frac{\pi_t(1 - \tau)}{1 + \pi_t}\right)(1 - \delta) = \beta^{-1}\]

increase in inflation requires decrease in \(f_k\) implying a higher steady state capital stock

– FO condition on bonds

\[
\frac{(1 + (1 - \tau) i_{t-1})}{1 + \pi_t} = \beta^{-1}
\]

increase in inflation requires a more than proportionate increase in the nominal interest rate