Money in the Utility Function
Walsh Chapter 2
1 Model

- Utility

\[ u(c_t, m_t) \text{ where } m_t = \frac{M_t}{P_tN_t} \]

- Output

\[ y_t = f(k_{t-1}) \]

- Budget constraint – small letters are real per capita – simplify by letting rate of growth of \( N \) (\( n \)) equal zero
\[ \omega_t = f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \left[ \frac{(1 + i_{t-1}) B_{t-1} + M_{t-1}}{P_t N_t} \right] \left[ \frac{P_{t-1}}{P_{t-1}} \right] \]
\[ = c_t + k_t + m_t + b_t \]

\[ \omega_t = f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \left[ \frac{(1 + i_{t-1}) b_{t-1} + m_{t-1}}{1 + \pi_t} \right] \]
\[ = c_t + k_t + m_t + b_t \]

where \( \frac{1}{1 + \pi_t} = \frac{P_{t-1}}{P_t} \)
- Optimization – Value function

\[ V(\omega_t) = \max \{ u(c_t, m_t) + \beta V(\omega_{t+1}) \} \]

s.t. budget constraint taken forward one period

\[ \omega_{t+1} = f(k_t) + \tau_{t+1} + (1 - \delta) k_t + \left[ \frac{(1 + i_t) b_t + m_t}{1 + \pi_{t+1}} \right] \]

and

\[ k_t = \omega_t - (c_t + m_t + b_t) \]
• First order necessary conditions

- \( c_t \)

\[
uc(c_t, m_t) - \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] = 0
\]

- \( b_t \)

\[
\beta V_\omega(\omega_{t+1}) \left[ -f_k(k_t) - (1 - \delta) + \frac{1 + it}{1 + \pi_{t+1}} \right] = 0
\]

- \( m_t \)

\[
u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1}) \left[ -f_k(k_t) - (1 - \delta) + \frac{1}{1 + \pi_{t+1}} \right] = 0
\]

- envelope theorem—take derivative of value function with respect to \( \omega_t \)

\[
V_\omega(\omega_t) = \beta V_\omega(\omega_{t+1}) [f_k(k_t) + 1 - \delta] = uc(c_t, m_t) \equiv \lambda_t
\]
• Transversality conditions

\[ \lim_{t \to \infty} \beta^t \lambda_t x_t = 0 \]  where \( x_t = k_t, b_t, m_t \)

• Results

– Capital Euler equation: use FO condition on \( c_t \) and \( V_\omega (\omega_{t+1}) = u_c (c_{t+1}, m_{t+1}) \)

\[ u_c (c_t, m_t) - \beta [f_k (k_t) + 1 - \delta] u_c (c_{t+1}, m_{t+1}) = 0 \]

– FO condition on \( b_t \) implies MPK plus depreciated capital equals real interest rate

\[ f_k (k_t) + (1 - \delta) = \frac{1 + i_t}{1 + \pi_{t+1}} \equiv 1 + r_t \]
– Bond Euler equation:

\[ u_c(c_t, m_t) - \beta \frac{1 + i_t}{1 + \pi_{t+1}} u_c(c_{t+1}, m_{t+1}) = 0 \]

– FO condition on money and envelope theorem yield money demand equation

\[ u_m(c_t, m_t) + \beta \left[ -f_k(k_t) - (1 - \delta) + \frac{1}{1 + \pi_{t+1}} \right] u_c(c_{t+1}, m_{t+1}) = 0 \]

\[ u_m(c_t, m_t) - u_c(c_t, m_t) + \frac{\beta u_c(c_{t+1}, m_{t+1})}{1 + \pi_{t+1}} = 0 \]

\[ \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = 1 - \left( \frac{\beta}{1 + \pi_{t+1}} \right) \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \]
\[ \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = 1 - \left( \frac{\beta}{1 + \pi_{t+1}} \right) \left( \frac{1 + \pi_{t+1}}{\beta (1 + i_t)} \right) = \frac{i_t}{1 + i_t} \]

- * hold 1 less unit of money

* its value next period is \( \frac{i}{1+\pi} \)

* present value next period is \( \frac{i}{(1+\pi)(1+r)} = \frac{i}{1+i} \)

money demand depends on nominal interest rate
2 Steady State Equilibrium

2.1 Definition

- Satisfy FO conditions
- Budget constraints
- Exogenous rate of growth of money $= \theta$
- All real variables are constant, implying $\pi = \theta$
2.2 Implications

2.2.1 Steady state capital stock

- Capital Euler equation with real variables constant implies

\[ \beta [f_k (k^{ss}) + 1 - \delta] = 1 \]

- Cobb-Douglas production function

\[ f (k) = k^\alpha \quad f_k (k) = \alpha k^{\alpha-1} \]

- Together imply capital in steady state is independent of money or money growth

\[ k^{ss} = \left( \frac{\alpha \beta}{1 + \beta (\delta - 1)} \right)^{\frac{1}{1-\alpha}} \]
2.2.2 Steady state value of transfers

- No government bonds and no government spending implies transfers must equal money growth

\[ \tau_t = \frac{M_t - M_{t-1}}{P_t} = m_t - m_{t-1} \frac{P_{t-1}}{P_t} = m_t - \frac{m_{t-1}}{1 + \pi_t} \]

in steady state \( m_t = m_{t-1} = m^{ss} \)

\[ \tau^{ss} = m^{ss} \left( \frac{\pi^{ss}}{1 + \pi^{ss}} \right) = m^{ss} \left( \frac{\theta}{1 + \theta} \right) \]

- Note that monetary policy (money growth) is not independent of fiscal policy (transfers)
2.2.3 Steady state consumption

- Aggregate budget constraint over agents where aggregate $b = 0$

$$f(k_{t-1}) + \tau_t + (1 - \delta) k_{t-1} + \frac{m_{t-1}}{1 + \pi_t} = c_t + k_t + m_t$$

transfers cancel with money terms, implying steady state consumption is independent of money or money growth

$$c^{ss} = f(k^{ss}) - \delta k^{ss}$$
2.2.4 Nominal rate of interest

- Capital Euler equation

\[ f_k (k^{ss}) + 1 - \delta = \frac{1}{\beta} \]

- First order condition on bonds

\[ f_k (k^{ss}) + (1 - \delta) = \frac{1 + i^{ss}}{1 + \pi^{ss}} \]

- Together

\[ \frac{1 + \pi^{ss}}{\beta} = 1 + i^{ss} \]
• Inflation raises the nominal interest rate

• Real interest rate is independent of inflation

\[
\frac{1}{\beta} = \frac{1 + i^{ss}}{1 + \pi^{ss}} \equiv 1 + r^{ss}
\]
2.2.5 Neutrality and superneutrality

- Money is neutral if a change in money has no effect on real variables
  - a change in the level of money has no effect on any real variable in the long-run implying that money is neutral in the long run in this model

- Money is superneutral if a change in the rate of growth of money has no effect on real variables, excepting real money
  - a change in the rate of growth of money affects the nominal interest rate and real money
  - it affects no other real variables
  - typically say money is superneutral in the long run in this model
2.3 Existence of equilibrium

- Assume marginal utility of money becomes zero or negative for all values of real money above a finite level.

- Separable utility

\[ u(c, m) = v(c) + \phi(m) \]

where \( \phi_m(m) \leq 0 \) for all \( m > \) some finite value.

- Equilibrium real money is given by

\[ \phi_m(m_{ss}) = \frac{i_{ss}}{1 + i_{ss}v_c(c_{ss})} \]

- For low values of \( m \), \( \phi_m(m) \) is large and above \( \frac{i_{ss}}{1 + i_{ss}v_c(c_{ss})} \).
– $\phi_{mm}(m) <$ such that as $m$ increases, left side eventually reaches right side

- Non-separable utility

\[ u_m(c^{ss}, m^{ss}) = \frac{i^{ss}}{1 + i^{ss}} u_c(c^{ss}, m^{ss}) \]

– as $m^{ss}$ increases, left-hand side falls, and if $u_{cm} < 0$, right side falls too

– could have no equilibria

– or multiple equilibria
2.4 Dynamics and Ruling out Explosive Equilibria

- money Euler equation in steady state with separable utility

\[ \phi_m (m_t) + \beta \left[ -\frac{1 + i_t}{1 + \pi_{t+1}} + \frac{1}{1 + \pi_{t+1}} \right] v_c (c^{ss}) = 0 \]

- using steady state relationship

\[ \frac{1 + i_t}{1 + \pi_{t+1}} \equiv 1 + r_t = \frac{1}{\beta}, \]

\[ \phi_m (m_t) + \frac{\beta v_c (c^{ss})}{1 + \pi_{t+1}} = v_c (c^{ss}) \]
• Write as a difference equation in $m$ by first multiplying both sides by real money

$$\frac{M_t}{P_t} \left( \frac{\beta v_c(c^{ss})}{1 + \pi_{t+1}} \right) = \frac{M_t}{P_t} (v_c(c^{ss}) - \phi_m(m_t))$$

• Use

$$\frac{M_t}{P_t} = \frac{M_{t+1} (1 + \pi_{t+1})}{P_{t+1} (1 + \theta)} = m_{t+1} \left( \frac{1 + \pi_{t+1}}{1 + \theta} \right)$$

• Money Euler equation becomes

$$B(m_{t+1}) = m_{t+1} \frac{\beta v_c}{1 + \theta} = m_t (v_c(c^{ss}) - \phi_m(m_t)) = A(m_t)$$

• Unstable system
– Rule out forever falling inflation because real money would become so large that violate transversality condition

– Can rule out rising inflation if

* \( \lim_{m \to 0} m_t \phi_m (m_t) > 0 \) because money would become negative next period.

* if \( \lim_{m \to 0} m_t \phi_m (m_t) = 0 \), cannot rule out equilibrium in which money eventually worthless - multiple equilibria
3 Interest Elasticity of Money Demand

- Let utility be CES

\[ u(c_t, m_t) = \left[ ac_t^{1-b} + (1 - a) m_t^{1-b} \right]^{\frac{1}{1-b}} \]

- Money demand

\[ \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} = \frac{1 - a}{a} \left( \frac{c_t}{m_t} \right)^b = \frac{i_t}{1 + i_t} \]

- Solving for \( m_t \)

\[ m_t = \left( \frac{1 - a}{a} \right)^{\frac{1}{b}} \left( \frac{i_t}{1 + i_t} \right)^{-\frac{1}{b}} c_t \]
• Transactions variable is consumption, not income

Taking logs

\[ \log m_t = \frac{1}{b} \log \left( \frac{1 - a}{a} \right) - \frac{1}{b} \log \left( \frac{i_t}{1 + i_t} \right) + \log c_t \]

• Empirically find money demand is interest inelastic \( \frac{1}{b} < 1 \), such that an increase in nominal interest increases expenditures on money

• Special case as \( b \to 1 \)

\[ \left( \frac{i_t}{1 + i_t} \right) m_t = \left( \frac{1 - a}{a} \right) c_t \]

Expenditures on money are proportional to consumption
4 Optimal Rate of Growth of Money

4.1 Friedman

- Steady state rate of growth of money equals rate of inflation and Fisher relation holds

- Optimization problem
  - Maximize
    \[ u(c^{ss}, m^{ss}) \]
  subject to
  \[ c^{ss} = f(k^{ss}) - \delta k^{ss} \]
First order conditions imply
\[
    u_m = \frac{i}{1+i} u_c
\]

- Want \( m \) as high as possible since costless to produce so \( u_m = 0 \)
- Requires \( i = 0 \).

\[
    1 + i = (1 + r)(1 + \pi) = 1
\]

implies that
\[
    \pi \approx -r
\]

- Optimal rate of inflation and money growth are both negative

- Major conflict with actual policy which seeks positive but small inflation
4.2 Inflation as a tax

- As price increases, real money balances fall – agents want to replace them – government prints money and buys real goods and services, allowing agents to replace real money balances – generates revenue for government

- Distortionary tax since leads agents to reduce real balances

- As a source of revenue can replace other distortionary taxes
4.3 Welfare cost of inflation

- Measure area under money demand curve, with money demand as a function of nominal interest

- Lucas finds about 1% of consumption for $i = .1$
5 Eliminating superneutrality

- Labor supply endogenous and utility not separable in money versus consumption and leisure
  - Increase in $\theta$ reduces $m$
  - Marginal utility of consumption relative to marginal utility of leisure depends on $m$

- Money is a factor of production
  - Effects depend on whether money is a substitute or complement for capital
• Taxes are levied on nominal values, not real values, implying that higher inflation yields higher real tax rates

- Real interest rate with taxes

\[
\frac{1 + i(1 - \tau)}{1 + \pi} = \frac{1 + [(1 + r)(1 + \pi) - 1](1 - \tau)}{1 + \pi}
\]

- Without taxes on interest, inflation does not affect the real interest rate

- With taxes on interest, an increase in inflation reduces the after-tax real interest rate

\[
\frac{\partial}{\partial \pi} \left( \frac{1+i(1-\tau)}{1+\pi} \right) = \frac{-\tau}{(1 + \pi^2)} < 0
\]
6 Dynamics in a MIU model

\[
    u(c_t, m_t, l_t) = \frac{[ac_t^{1-b} + (1 - a)m_t^{1-b}]^{1-\phi}}{1-\phi} + \psi \frac{l_t^{1-\eta}}{1-\eta}
\]

\[
    u_c(c_t, m_t, l_t) = \frac{[ac_t^{1-b} + (1 - a)m_t^{1-b}]^{\frac{b-\phi}{1-b}}}{1-b} a (1 - b) c^{-b}
\]

\[
    u_{cm}(c_t, m_t, l_t) = (b - \phi) \left[ ac_t^{1-b} + (1 - a)m_t^{1-b} \right]^{\frac{b-\phi}{1-b}-1} a (1 - b) c^{-b} (1 - b) (1 - a) m_t^{-b}
\]

- Sign of \( u_{cm}(c_t, m_t, l_t) \) depends on sign of \( (b - \phi) \)
• An increase in inflation reduces $m_t$

  – If $u_{cm}(c_t, m_t, l_t) < 0$, reduction in money increases marginal utility of consumption requiring a reduction in consumption

  – Since $\frac{u_l}{u_c} = MPL$, less leisure and more work

  – Output is higher, consumption is lower, and the approach to steady state is faster

  – Expected inflation has real effects

• Since putting money in the utility function is a short-cut, we know nothing about $u_{cm}(c_t, m_t, l_t)$
7 Summary

7.1 Positive aspects of MIU model

- Simple to implement in models

- Yields money demand function of a form typically supported empirically

- Straightforward welfare implications
7.2 Negative aspects of MIU model

- Money does not yield utility – short-cut for something else

- Difficulties
  - What is sign of \( u_{cm} \)?
  - What is \( \lim_{m_t \to 0}(v_c - \phi_m)m_t \)?

- Optimality calls for \( i = 0 \), and negative money growth – very different from actual monetary policy