Small Open Economy RBC Model
Uribe, Chapter 4
1 Basic Model

1.1 Uzawa Utility

\[ E_0 \sum_{t=0}^{\infty} \theta_t U(c_t, h_t) \]

\[ \theta_0 = 1 \]

\[ \theta_{t+1} = \beta(c_t, h_t) \theta_t; \quad \beta_c < 0; \quad \beta_h > 0. \]

- Time-varying discount factor
– With a constant discount factor, consumption and net foreign assets are non-stationary

* Rise permanently with an increase in productivity

* Unit roots imply no steady state

* Cannot claim that the linearized model, which converges to a steady state, behaves as the original model

* Cannot compute unconditional moments
– Discount factor is decreasing in wealth

* As wealth increases, $c$ increases and $h$ decreases reducing $\theta_{t+1}$ below $\theta_t$

* Equivalently, agents become less patient as wealth increases

* Positive productivity shock raises wealth and consumption and reduces hours

* Therefore, $\theta$ falls, increasing consumption, sending wealth back down

* Get a steady state value of wealth with this discount factor and therefore a steady state
1.2 Household Budget Constraint

- Write in terms of household debt \((d)\) instead of assets

\[
d_t = (1 + r_{t-1}) d_{t-1} - y_t + c_t + i_t + \Phi (k_{t+1} - k_t)
\]

- Cost of Investment includes capital adjustment costs \(\Phi (k_{t+1} - k_t)\)
  
  - Adjustment cost reduce the volatility of investment
  
  - Adjustment costs is increasing in the size of the change in capital, reducing the optimal size of investment
  
  - Restrictions assure that when capital is fixed in the steady state, there are no investment costs

\[
\Phi (0) = \Phi' (0) = 0
\]
• Output

\[ y_t = A_t F(k_t, h_t) \]

• Evolution of capital

\[ k_{t+1} = i_t + (1 - \delta) k_t \]
1.3 Optimization

- Households choose \( \{c_t, h_t, k_{t+1}, d_t, \theta_{t+1}\} \)
  
  - Subject to above constraints

  - and NPG condition requiring the present value of debt in the limit to be non-positive

  \[
  \lim_{j \to \infty} E_t \frac{d_{t+j}}{\prod_{s=1}^{j} (1 + r_s)} \leq 0
  \]

  - Substitute for investment in household budget constraint

  \[
  d_t = (1 + r_{t-1}) d_{t-1} - y_t + c_t + k_{t+1} - (1 - \delta) k_t + \Phi (k_{t+1} - k_t)
  \]
• Lagrange multipliers
  
  – $\theta_t \lambda_t$ on household budget constraint
  
  – $\eta_t$ on evolution of discount factor

\[
\theta_{t+1} = \beta (c_t, h_t) \theta_t
\]
1.4 First Order Conditions

- $d_t$

$$\lambda_t = \beta(c_t, h_t) E_t \lambda_{t+1} (1 + r)$$

- $\beta(c_t, h_t)$ is the ratio of two discount factors

$$1 = \frac{\beta(c_t, h_t) E_t \lambda_{t+1} (1 + r)}{\lambda_t}$$

- if $\beta(1 + r)$ is fixed at unity a wealth-reducing shock which raises $\lambda_t$
  raises $E_t \lambda_{t+1}$ equally, hence permanently

- with $\beta$ endogenous, the reduction in wealth reduces consumption which
  raises $\lambda_t$ and $\beta(c_t, h_t)$

* increase in $E_t \lambda_{t+1}$ is less than increase in $\lambda_t$
* eventually $\lambda_t$ returns to its steady-state value

- $c_t$

$$
\lambda_t = U_c(c_t, h_t) - \eta_t \beta_c(c_t, h_t)
$$

- Marginal utility of wealth ($\lambda_t$) equals marginal utility of consumption $[U_c(c_t, h_t)]$ less the marginal value of the reduction in the discount factor (positive term)

- Unit decline in discount factor in turn reduces utility in period $t$ by $\eta_t$
\( h_t \)

\[ U_h(c_t, h_t) + \lambda_t A_t F_h(k_t, h_t) - \eta_t \beta_h(c_t, h_t) = 0 \]

\[-[U_h(c_t, h_t) - \eta_t \beta_h(c_t, h_t)] = A_t F_h(k_t, h_t) [U_c(c_t, h_t) - \eta_t \beta_c(c_t, h_t)]\]

- marginal disutility of hours includes effect of increase in hours in increasing the discount factor

- adjusted marginal disutility of hours equals the wage times the adjusted marginal utility of consumption
\[ \eta_t = -E_t U(c_{t+1}, h_{t+1}) + E_t \eta_{t+1} \beta(c_{t+1}, h_{t+1}) \]

- Solving forward

\[ \eta_t = -E_t \sum_{j=1}^{\infty} \frac{\theta_{t+j}}{\theta_{t+1}} U(c_{t+j}, h_{t+j}) \]

- \( \eta_t \) is the negative of the present discounted value of utility
\( k_{t+1} \)

\[
\begin{align*}
\lambda_t \left[ 1 + \Phi'(k_{t+1} - k_t) \right] \\
= \beta(c_t, h_t) E_t \lambda_{t+1} \left[ A_{t+1} F_k(k_{t+1}, h_{t+1}) + 1 - \delta + \Phi'(k_{t+2} - k_{t+1}) \right]
\end{align*}
\]

– cost of one additional unit of capital includes adjustment costs

– benefits are discounted utility value of capital's marginal product plus its undepreciated value plus saved adjustment cost going forward since capital it higher
1.5 Assumptions

- Perfect capital mobility such that interest rate in small open economy equals fixed world rate

\[ r_t = r \]

- Productivity is stationary and AR(1)

\[ \ln A_{t+1} = \rho \ln A_t + \epsilon_{t+1} \quad \rho \in (-1, 1) \quad \epsilon \sim N \left(0, \sigma_{\epsilon}^2\right) \]

- Functional forms

\[ U (c, h) = \frac{\left[ c - \omega^{-1} h \omega \right]^{1-\gamma} - 1}{1 - \gamma} \]
\[ \beta (c, h) = \left[ 1 + c - \omega^{-1} h^\omega \right]^{-\psi_1} \]
\[ F(k, h) = k^\alpha h^{1-\alpha} \]
\[ \Phi(x) = \frac{\phi}{2}x^2 \quad \phi > 0 \]

- Combining FO conditions on \( c_t \) and \( h_t \), functional forms make labor supply independent of consumption (ratios of marginal utilities depend only on labor)

\[ A_t (1 - \alpha) k_t^\alpha h_t^{-\alpha} = h_t^{\omega-1} \]

- Lhs is wage at which firm is willing to hire labor, hence labor demand and rhs is wage as which household willing to supply hours, hence labor supply
2 Calibration to Canadian Economy (Annual)

- $\omega = 1.455$ implies a high labor supply elasticity of $\frac{1}{\omega - 1} = 2.2$

  - Totally differentiate second equation below with respect to wage and labor supply

  $A(1 - \alpha)k^\alpha h^{-\alpha} = h^{\omega - 1} = w_t$

  $$dw = (\omega - 1) h^{\omega - 2} dh$$

  $$\frac{dh}{dw} = \frac{1}{\omega - 1} h^{\omega - 2}$$

  $$\frac{wdh}{hdw} = \frac{1}{\omega - 1}$$
• \( \delta = 0.1 \) implies that capital depreciates at 10% per year

• \( \alpha = .32 \) implies a capital income share of 32%

• \( r = .04 \) is the average real rate of return of broad measures of the stock market in developed countries post WWII

• \( \psi_1 \) is the elasticity of the discount factor with respect to the composite 
  \( 1 + c - \omega^{-1} h \omega \)
  
  – match average Canadian trade-balance-to-GDP ratio of 2%
  
  – steady state value of FO condition on bonds yields

  \[ \beta (c, h) (1 + r) = 1 \]
- steady state value of FO condition on capital with $A = 1$ yields

$$r + \delta = \alpha \left( \frac{h}{k} \right)^{1-\alpha}$$

$$\frac{k}{h} = \left( \frac{\alpha}{r + \delta} \right)^{\frac{1}{1-\alpha}}$$

implying that the steady-state capital labor ratio is independent of $\psi_1$

- steady state value of hours is also independent of $\psi_1$

$$h = \left[ (1 - \alpha) \left( \frac{k}{h} \right)^\alpha \right]^{\frac{1}{\omega-1}}$$

- steady-state values of capital, output, and investment ($i = \delta k$) will all be independent of $\psi_1$
– steady-state value of FO condition on bonds

\[ \beta (c, h) (1 + r) = \left[ 1 + c - \omega^{-1} h \right]^{-\psi_1} (1 + r) = 1 \]

* use resource constraint

\[ c = y - i - tb \]

to substitute for \( c \)

\[ \left[ 1 + y - i - tb - \omega^{-1} h \right]^{-\psi_1} (1 + r) = 1 \]

* solve for \( \frac{tb}{y} \)

\[ \frac{tb}{y} = 1 - \frac{i}{y} - \frac{(1 + r)^{\frac{1}{\psi_1}} + \omega^{-1} h - 1}{y} \]
– since hours, capital, investment, and output are independent of $\psi_1$, can calibrate $\psi_1$ to match $\frac{tb}{y}$

– find $\psi_1$ is increasing in $\frac{tb}{y}$

– dual role of $\psi_1$
  * determines steady-state trade balance/output ratio
  * governs speed of convergence to steady state
  * might want to separate these allowing speed of convergence to be small enough as to not significantly affect business cycle dynamics

* add a parameter and respecify $\beta(c_t, h_t)$

$$\beta(c_t, h_t) = \bar{\beta} \left[ 1 + (c_t - c) - \omega^{-1} (h_t^\omega - h^\omega) \right]^{-\psi_1}$$
where $c$ and $h$ are steady-state values

- $\phi = 0.028$ matches standard deviation of investment of 9.8
- $\sigma \epsilon = 0.0129$ matches standard deviation of output of 2.8
- $\rho = 0.42$ matches serial correlation of output of 0.61
3 Equilibrium

3.1 Definition

A competitive equilibrium is a set of processes \( \{d_t, c_t, h_t, y_t, i_t, k_{t+1}, \eta_t, \lambda_t, A_t\} \) satisfying the resource constraint and the first order conditions, given the fixed world interest rate, \( r \), initial conditions \( A_0, d_{-1} \), and \( k_0 \) and the exogenous process \( \{\epsilon_t\} \).
3.2 Approximation

- Solutions where endogenous variables fluctuate in small neighborhood around steady state
  
  - Debt will be bounded implying

  \[
  \lim_{j \to \infty} E_t d_{t+j} \left( \frac{1}{1 + r} \right)^j = 0
  \]

  - Write system as

  \[
  E_t f (x_{t+1}, x_t) = 0
  \]

  - Cannot solve non-linear system

- Linearize about steady state
– Express most variables as percent deviations about steady state

\[ \hat{w}_t \equiv \log \left( \frac{w_t}{w} \right) \approx \frac{w_t - w}{w} \]

– For variables that can take on negative values, like trade balance, or variables already expressed as percent, like interest rates, just use first difference

\[ \hat{w}_t \equiv w_t - w \]

– Linearized system is expressed as

\[ A\hat{x}_{t+1} = B\hat{x}_t \]

where \( A \) and \( B \) are conformable square matrices made up of known coefficients in the calibrated linearized model
● Ten variables in linearized system

  – state variables

    * variables whose \( t \) values are predetermined (determined before \( t \))

    * variables whose values are exogenous

    * include \( \hat{k}_t, \hat{d}_{t-1}, \hat{A}_t, \) and \( \hat{r} \)

  – co-state variables

    * endogenous variables have values not predetermined in period \( t \)
Initial conditions

- Three known initial conditions $\hat{k}_0, \hat{d}_{-1}, \hat{A}_0$

- Determine other initial conditions to satisfy boundedness condition

$$\lim_{j \to \infty} \left[ E_t \hat{x}_{t+j} \right] = 0$$
4 Model Performance

4.1 Successes

- Model is calibrated to match some moments and does well here by construction
  - volatility of output
  - volatility of investment,
  - serial correlation of output
• Volatility rankings
  – investment volatility greater than output volatility
  – consumption volatility less than output volatility

• trade balance is countercyclical
  – productivity shock increases investment more than it reduces savings due to consumption smoothing
  – result due to parameters $\phi$, governing cost of investment, and $\rho$, persistence of productivity shock and these values were calibrated independent of trade balance performance
4.2 Failures

- too little countercyclicality in trade balance
  
  - need investment and/or consumption to increase more in response to positive productivity shock

- correlation between
  
  - consumption and output too high
  
  - between hours and output is too high at unity
    * due to functional form for the period utility index
* condition for equilibrium hours

\[ A_t (1 - \alpha) k_t^\alpha h_t^{-\alpha} = h_t^{\omega - 1} \]

\[(1 - \alpha) y_t = h_t^\omega\]

* log-linearized version is

\[ \hat{y}_t = \omega \hat{h}_t \]

* implying a correlation of unity
5 Impulse responses

- Output, investment, hours, and consumption all respond positively to a productivity innovation.

- The trade balance/GDP and the current account/GDP both respond negatively.
5.1 Countercyclicality of the trade balance

- Investment must respond strongly enough
  - requires that adjustment cost not be too high
  - persistence of productivity must be high enough

- Consumption must respond strongly enough
  - requires persistence relatively high
6 Other Ways to Impose Stationarity

6.1 Debt-elastic interest rate

- Assume interest rate paid by small open economy is increasing in its aggregate external debt

\[ r_t = r + p(\tilde{d}_t) \]

- Introduces a risk-premium on debt

- Assume agent does not take into account the effect of his change in debt on the country risk premium implying that the Euler equation is

\[ \lambda_t = \beta (1 + r_t) \lambda_{t+1} \]
Dynamics of the model: begin away from steady state and show that end up in steady state

- Assume $\beta (1 + r_t) > 1$

- Agent are saving, wealth and consumption are rising and marginal utility $(\lambda)$ is falling

- Increase in wealth implies a reduction in foreign debt and a reduction in $r_t$, returning system to steady state

Can calibrate the $p$ parameter to be small enough that it yields stationarity without affecting business cycle dynamics
6.2 Portfolio Adjustment Costs

- Assume there is a cost to holding debt in quantity different from its long-run equilibrium value

\[ d_t = (1 + r_{t-1})d_{t-1} - y_t + c_t + k_{t+1} - (1 - \delta) k_t + \Phi (k_{t+1} - k_t) + \frac{\psi_3}{2} (d_t - \bar{d})^2 \]

reducing the marginal utility of debt due to the adjustment cost

- Euler equation becomes

\[ \lambda_t [1 - \psi_3 (d_t - \bar{d})] = \beta (1 + r) E_t \lambda_{t+1} \]

- Dynamics of the model: shock model away from steady state and show that end up in steady state
– Assume $d_t$ increases from steady state value such that $d_t > \bar{d}$

$$[1 - \psi_3 (d_t - \bar{d})] = \beta (1 + r) \frac{E_t \lambda_{t+1}}{\lambda_t}$$

– Lhs decreases below unity requiring

$$\frac{E_t \lambda_{t+1}}{\lambda_t} < 1$$

– the increase in debt is a reduction in wealth such that consumption falls and marginal utility rises

– adjustment costs require that $E_t \lambda_{t+1}$ rise by less than $\lambda_t$ rises, requiring that current consumption fall by more than future consumption

– agent begins saving to replace wealth lost by shock returning system to steady state
• Can calibrate $\psi_3$ to be small enough that it yields stationarity without affecting business cycle dynamics much
6.3 Precautionary Savings

- Euler equation with log utility

\[
\frac{1}{c_t} = \beta (1 + r) \frac{1}{E_t} \frac{1}{c_{t+1}} > \beta (1 + r) \frac{1}{E_t c_{t+1}}
\]

- With $\beta (1 + r) = 1$

\[
E_t c_{t+1} > c_t
\]

  - Such that consumption is rising implying that wealth is growing
  
  - Increase in wealth reduces magnitude of precautionary motive implying wealth is rising at falling rate
  
  - Eventually reach a steady state
- However, since world cannot have steady state at $\beta (1 + r) = 1$, really need to consider $\beta (1 + r) < 1$ to reduce world savings

- Problem
  - Precautionary saving requires a positive third derivative of utility
  - Cannot use in a linearized model because linearization eliminates the third derivative
7 Business Cycles in Emerging Markets (Aguiar and Gopinath 2007)

7.1 Model

- Utility per capita

\[
E_0 \sum_{t=0}^{\infty} \beta^t \frac{C_t^\gamma (1 - h_t)^{1-\gamma}}{1-\sigma} - 1
\]

- Budget constraint per capita

\[
\frac{D_{t+1}}{R_t} = \frac{D_t + C_t + K_{t+1} - (1 - \delta) K_t + \frac{\phi}{2} \left( \frac{K_{t+1}}{K_t} - g \right)^2 K_t - A_t K_t^\alpha (X_t h_t)^{1-\alpha}}{R_t}
\]
First order conditions for household letting $\Lambda_t$ denote multiplier

- $C_t$
  \[
  \left[ C_t^\gamma (1 - h_t)^{1-\gamma} \right]^{-\sigma} \gamma C_t^{\gamma - 1} (1 - h_t)^{1-\gamma} - \Lambda_t = 0
  \]

- $h_t$
  \[
  - \left[ C_t^\gamma (1 - h_t)^{1-\gamma} \right]^{-\sigma} (1 - \gamma) C_t^\gamma (1 - h_t)^{-\gamma} + \Lambda_t (1 - \alpha) A_t K_t^\alpha X_t^{1-\alpha} h_t^{-\alpha}
  \]

- $D_{t+1}$
  \[
  \Lambda_t \frac{1}{R_t} - E_t \Lambda_{t+1} \beta = 0
  \]
\[ - K_{t+1} \]

\[ \Lambda_t \left( 1 + \phi \left( \frac{K_{t+1}}{K_t} - g \right) \right) \]

\[ = \beta E_t \Lambda_{t+1} [1 - \delta + \alpha A_{t+t} K_{t+1}^{\alpha-1} (X_{t+1} h_{t+1})^{1-\alpha} \]

\[ + \phi \left( \frac{K_{t+2}}{K_{t+1}} - g \right) \frac{K_{t+2}}{K_{t+1}} - \phi \left( \frac{K_{t+2}}{K_{t+1}} - g \right)^2 ] \]
• Stationary productivity shock

\[ \ln A_t = \rho_a \ln A_{t-1} + \sigma_a \epsilon_t^a \quad 0 < \rho_a < 1 \]

• Non-Stationary productivity shock

  – Define

  \[ g_t = \frac{X_t}{X_{t-1}} \]

  – Growth is stationary

  \[ \ln (g_t - g) = \rho_g \ln (g_{t-1} - g) + \sigma_g \epsilon_t^g \quad 0 < \rho_g < 1 \quad g > 1 \]

  – Unit root in productivity gives other variables unit root, so equilibrium is not stationary
• Find a way to transform the model to make transformed variables stationary
7.2 Transformation to make the model stationary

- Divide trending variables by $X_{t-1}$ and define resulting variables, like $\frac{C_t}{X_{t-1}} \equiv c_t$

- In FO condition on $C_t$, expressing in terms of $c_t$ requires multiplying equation by $X_{t-1}^{1+\gamma(\sigma-1)}$
  
  - Therefore, define

  $$\lambda_t = \Lambda_t X_{t-1}^{1+\gamma(\sigma-1)}$$

- Detrended FO conditions
\[-c_t \\quad \left[ c_t^\gamma (1 - h_t)^{1-\gamma} \right]^{-\sigma} \gamma c_t^\gamma (1 - h_t)^{1-\gamma} - \lambda_t = 0 \]

\[-h_t \text{ multiply equation by } X_t^\gamma(\sigma-1) \]
\[
\left[ c_t^\gamma (1 - h_t)^{1-\gamma} \right]^{-\sigma} (1 - \gamma) c_t^\gamma (1 - h_t)^{-\gamma} = \lambda_t (1 - \alpha) A_t k_t^\alpha g_t^{1-\alpha} h_t^{-\alpha} \]

\[-\text{ Using the two equations together} \]
\[
\frac{(1 - \gamma) c_t}{\gamma (1 - h_t)} = (1 - \alpha) A_t g_t \left( \frac{k_t}{g_t h_t} \right)^\alpha \]

\[-D_{t+1} \text{ multiply equation by } X_t^{1+\gamma(\sigma-1)} \]
\[
\lambda_t \frac{1}{R_t} - \beta E_t \Lambda_t X_t^{1+\gamma(\sigma-1)} = 0 \]
\[
\lambda_t \frac{1}{R_t} - \beta E_t \lambda_{t+1} g_t^{1-\gamma(\sigma-1)} = 0
\]

\[
- K_{t+1}
\]

\[
\frac{K_{t+2}}{K_{t+1}} = \frac{K_{t+2} X_{t+1}}{X_t X_{t+1}} = \frac{g_{t+1} k_{t+2}}{k_{t+1}}
\]

\[
\lambda_t \left( 1 + \phi \left( \frac{g_t k_{t+1}}{k_t} - g \right) \right)
\]

\[
= \beta g_t^{1-\gamma(\sigma-1)} E_t \lambda_{t+1} [1 - \delta + \alpha A_{t+1} \left( \frac{k_{t+1}}{g_{t+1} h_{t+1}} \right)^{\alpha-1}
\]

\[
+ \phi \left( \frac{g_{t+1} k_{t+2}}{k_{t+1}} - g \right) \frac{g_{t+1} k_{t+2}}{k_{t+1}} \right] \frac{1}{2} \left( \frac{g_{t+1} k_{t+2}}{k_{t+1}} - g \right)^2
\]

\[
- \text{Linearized system has the property that a transitory shock will create a permanent increase in consumption if } R_t \text{ is exogenous}
\]
• Risk premium on interest rate increasing in debt above some minimum level, $\bar{d}$

$$R_t = R^* + \psi \left[ e^{d_{t+1} - \bar{d}} - 1 \right]$$
7.3 Equilibrium

7.4 Properties of equilibrium

- Stationary in detrended variables

- Actual variables inherit the unit root in $X_t$ since

$$C_t = c_t X_{t-1}$$

- Stationary distribution of wealth around its mean due to assumption about risk-premium on interest rate
7.5 Calibration

- Calibrate $\beta = 0.98$, $\gamma = 0.36$, $\psi = 0.001$, $\alpha = 0.68$, $\sigma = 2$, $\delta = 0.05$

- Estimate parameters for stochastic processes and adjustment cost $\phi$ at quarterly horizon using Mexico 1980Q1 to 2003Q1.

  - $\sigma_g = 0.0213$, $\rho_g = 0.00$, $g = 1.0066$
  
  - $\sigma_a = 0.0053$, $\rho_a = 0.95$

  - $\phi = 1.37$

- What fraction of variance of Solow residual is determined by the permanent shock?
- Detrended output

\[ A_t k_t^\alpha (g_t h_t)^{1-\alpha} \]

- Solow Residual

\[ SR_t = A_t g_t^{1-\alpha} \]

- Fraction of variance of Solow residual accounted for by non-stationary shock is 88%

- Compares to 40% for Canada

- Implies that importance of non-stationary shock is major difference in rich and emerging market countries

- Potential problem
- Need long time series to confidently identify a data series as non-stationary
7.6 Compare moments to data

- Since data is non-stationary, need to make it stationary to compute moments
  - Unconditional mean and variance do not exist in non-stationary data
  - Use HP filter
  - Theoretically should estimate the non-stationary trend and deflate by that as in model

- Consumption is more volatile than output
  - Importance of non-stationary output
• Trade balance is countercyclical

• Other matches also good