Nominal Exchange Rates
Obstfeld and Rogoff, Chapter 8
1 Cagan Model of Money Demand

1.1 Money Demand

- Demand for real money balances \((\frac{M}{P})\) depends negatively on expected inflation.

  - In logs

  \[
m_t^d - p_t = -\eta E_t [p_{t+1} - p_t]
  \]

- Cagan worked on hyperinflation
– Changes in output likely to be negligible compared with changes in inflation

– Nominal interest rate equals real rate plus inflation from the Fisher relation

\[ 1 + i_{t+1} = (1 + r_{t+1}) \frac{P_{t+1}}{P_t} \]

– Changes in real rate are also negligible
1.2 Money Supply

- Supply of nominal money is set exogenously

1.3 Money Market Equilibrium

- Money demand must equal money supply

\[ m^d_t = m_t \]

\[ m_t - p_t = -\eta E_t [p_{t+1} - p_t] \]
• Assume perfect foresight with no stochastic variables

  – Solve for the current price as a function of future price

    \[ p_t = \frac{1}{1 + \eta}m_t + \frac{\eta}{1 + \eta}p_{t+1} \]

  – Taking forward one period

    \[ p_{t+1} = \frac{1}{1 + \eta}m_{t+1} + \frac{\eta}{1 + \eta}p_{t+2} \]

  – Substituting into the expression for \( p_t \) yields

    \[ p_t = \frac{1}{1 + \eta}m_t + \frac{\eta}{1 + \eta} \left[ \frac{1}{1 + \eta}m_{t+1} + \frac{\eta}{1 + \eta}p_{t+2} \right] \]
- Iterate forward and solve for current price

\[
p_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} m_s + \lim_{T \to \infty} \left( \frac{\eta}{1 + \eta} \right)^T p_{t+T}
\]

- Assume that the price level grows more slowly that the inverse of \( \frac{\eta}{1+\eta} \) (necessary to eliminate speculative bubbles)

\[
p_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} m_s
\]
– Special cases

* money is constant forever at $m_t$

$$p_t = \frac{1}{1+\eta} \left[ 1 + \frac{\eta}{1+\eta} + \left( \frac{\eta}{1+\eta} \right)^2 + \ldots \right] m_t$$

$$= \frac{1}{1+\eta} \left[ \frac{1}{1-\frac{\eta}{1+\eta}} \right] m_t = m_t$$

· price is equal to the money supply

· a permanent change in money changes price proportionately

· monetary neutrality: a permanent change in money changes price proportionately and affects nothing real
* constant money growth

\[ m_t = m_{t-1} + \mu \]

\[ m_s = m_t + \mu (s - t) \]

- substitute and solve the infinite sum to yield

\[ p_t = m_t + \eta \mu \]

- future monetary growth raises the current price level (why?)
* announcement at time $t$ that money supply will rise permanently on date $T > t$

$$p_t = m_t$$

$$+ \frac{1}{1+\eta} \left[ \left( \frac{\eta}{1+\eta} \right)^{T-t} \left( 1 + \frac{\eta}{1+\eta} + \left( \frac{\eta}{1+\eta} \right)^2 + \ldots \right) \right] \times (m_T - m_t)$$

$$= m_t + \left( \frac{\eta}{1+\eta} \right)^{T-t} (m_T - m_t)$$

- price level will begin to rise on the date of the announcement and continue to rise until it reaches its long-run equilibrium value at date $T$
• Stochastic money supply

  – No bubbles solution

\[ p_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} E_t m_s \]

  – Assume money supply is AR(1)

\[ m_t = \rho m_{t-1} + \epsilon_t \quad 0 \leq \rho < 1 \]

where mean of \( \epsilon_t \) is zero, so that

\[ E_t \epsilon_{t+i} = 0 \]
- Taking expectations and substituting into the solution

\[
p_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta \rho}{1 + \eta} \right)^{s-t} \quad m_t = \frac{m_t}{1 + \eta} \left( \frac{1}{1 - \frac{\eta \rho}{1 + \eta}} \right)
\]

\[
= \frac{m_t}{1 + \eta (1 - \rho)}
\]
1.4 Seigniorage

- Seigniorage is the real revenue the government acquires by printing money

\[
\text{Seigniorage} = \frac{M_t - M_{t-1}}{P_t} = \left( \frac{M_t - M_{t-1}}{M_t} \right) \frac{M_t}{P_t}
\]

- A higher rate of growth of money reduces money demand, mitigating the increase in seigniorage

- The Cagan money demand function has a revenue maximizing rate of money growth
Money demand in levels is

\[
\frac{M_t}{P_t} = \left( \frac{P_{t+1}}{P_t} \right)^{-\eta}
\]

If money growth is \( \mu \), then inflation is also \( \mu \)

\[
1 + \mu = \frac{M_t}{M_{t-1}} = \frac{P_t}{P_{t-1}}
\]

* Use this expression to compute

\[
\frac{M_t - M_{t-1}}{M_t} = \mu \frac{M_{t-1}}{M_t} = \frac{\mu}{1 + \mu}
\]

Seigniorage

\[
\frac{\mu}{1 + \mu} (1 + \mu)^{-\eta} = \mu (1 + \mu)^{-1-\eta}
\]
maximize seigniorage with respect to $\mu$ to yield

$$(1 + \mu)^{-1-\eta} - \mu (1 + \eta)(1 + \mu)^{-2-\eta} = 0$$

$$1 - \mu (1 + \eta)(1 + \mu)^{-1} = 0$$

$$\frac{1}{1 + \eta} = \frac{\mu}{1 + \mu}$$

$$\mu = \frac{1}{\eta}$$

* Why would a government ever exceed its seigniorage revenue maximizing rate of money growth as governments do in hyperinflation?

* If agents do not believe the government can commit to a particular rate of money growth, money demand will be lower, and the seigniorage revenue maximizing rate of money growth will be lower
1.5 Monetary Model of the Exchange Rate

Building Blocks

- Money demand
  - real money demand depends negatively on nominal interest and positively on real output
    \[ m_t - p_t = -\eta i_{t+1} + \phi y_t \]
  - where variables are logs except that \( i = \log (1 + i) \approx i \)
- Purchasing power parity
  - one world good
  - arbitrage assures PPP
  \[ P = \mathcal{E} P^* \]
  - PPP in logs
  \[ p = e + p^* \]
• Interest rate parity

  – gross return on domestic bonds

    \[ 1 + i_{t+1} \]

  – gross return on foreign bonds

    * take 1 unit of domestic currency and purchase \( \frac{1}{\mathcal{E}_t} \) units of foreign currency

    * use that to buy foreign bonds such that at end of period have \( \left(1 + i^*_{t+1}\right) / \mathcal{E}_t \) units of foreign currency

    * will sell those for \( \mathcal{E}_{t+1} \) units of domestic currency, implying that expected gross return on foreign bonds is

    \[
    \left(1 + i^*_{t+1}\right) \frac{E_t \mathcal{E}_{t+1}}{\mathcal{E}_t}
    \]
- arbitrage, ignoring risk, requires that expected returns on the two assets be equal if both are held in equilibrium, yielding IRP

\[ 1 + i_{t+1} = \left(1 + i^*_{t+1}\right) \frac{E_t \mathcal{E}_{t+1}}{\mathcal{E}_t} \]

- in logs

\[ i_{t+1} = i^*_{t+1} + E_t e_{t+1} - e_t \]

where we ignore the inconvenient fact that \( \log E_t \mathcal{E}_{t+1} \neq E_t \log \mathcal{E}_{t+1} \)
Monetary model of the exchange rate

- Small open economy

  - Substitute PPP and IRP into money demand
    \[
    (m_t - \phi y_t + \eta i^*_{t+1} - p^*_t) - e_t = -\eta (E_t e_{t+1} - e_t)
    \]
  
  - Equation is formally identical to Cagan’s money demand equation if we treat \((m_t - \phi y_t + \eta i^*_{t+1} - p^*_t)\) as an exogenous stochastic process
  
  * Solve for exchange rate
    \[
    e_t = \frac{1}{1 + \eta} \left[ m_t - \phi y_t + \eta i^*_{t+1} - p^*_t + E_t e_{t+1} \right]
    \]
* Solving forward imposing the zero limit term

\[ e_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} E_t \left( m_s - \phi y_s + \eta i_{s+1}^* - p_s^* \right) \]

* higher path for the money supply depreciates domestic currency

* higher path for output raises money demand and appreciated domestic currency
• Two-country model

  – Add foreign money demand to the model

  \[ m_t^* - p_t^* = -\eta i_{t+1}^* + \phi y_t^* \]

  – Solve domestic and foreign money demand for price and subtract

  \[ p_t = m_t + \eta i_{t+1} - \phi y_t \]

  \[ p_t^* = m_t^* + \eta i_{t+1}^* - \phi y_t^* \]

  \[ p_t - p_t^* = m_t - m_t^* + \eta \left( i_{t+1} - i_{t+1}^* \right) - \phi \left( y_t - y_t^* \right) \]

  – Substitute PPP and IRP

  \[ e_t = m_t - m_t^* + \eta \left( E_t e_{t+1} - e_t \right) - \phi \left( y_t - y_t^* \right) \]
- Solve for exchange rate imposing the zero limit term

\[ e_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} E_t (m_s - m_s^* - \phi (y_s - y_s^*)) \]
• Exchange rates are an asset price, subject to the effects of changes in expectations

  – Assume the only stochastic variable is domestic money, and mean of all others is zero

  – Money supply is non-stationary and has a stationary rate of growth

    \[ m_t - m_{t-1} = \rho (m_{t-1} - m_{t-2}) + \epsilon_t \]

  – Solve for exchange rate by leading one period and subtracting

    \[ E_t e_{t+1} - e_t = \frac{1}{1 + \eta} \sum_{s=t}^{\infty} \left( \frac{\eta}{1 + \eta} \right)^{s-t} E_t (m_{s+1} - m_s) \]
– Substituting for the time series process for money growth and using earlier results yields

$$E_t e_{t+1} - e_t = \frac{\rho (m_t - m_{t-1})}{1 + \eta (1 - \rho)}$$

– Substituting for the expected rate of change of the exchange rate into the monetary model yields

$$m_t - e_t = -\eta (E_t e_{t+1} - e_t) = \frac{-\eta \rho (m_t - m_{t-1})}{1 + \eta (1 - \rho)}$$
- Solving for the exchange rate yields

\[ e_t = m_t + \frac{\eta \rho (m_t - m_{t-1})}{1 + \eta (1 - \rho)} \]

- Note that because of the time series process assumed for money, an increase in the money supply has a more than proportionate effect on the current exchange rate because an increase today implies higher expected future monetary growth.
• Empirical implementation of the model
  
  – Random walk model is better at prediction (Meese and Rogoff)
  
  – Exchange rate is much more volatile than fundamentals
  
  – Model fits better and predicts better at long horizons than at short
2 Money in the Utility Function

2.1 Objective function

- Utility depends positively on consumption and real money balances

\[
U_t = \sum_{s=t}^{\infty} \beta^{s-t} u \left( C_s, \frac{M_s}{P_s} \right) \quad \beta < 1
\]
• Why is money in the utility function?
  
  – We do not assume that money yields utility directly just as we never put wealth in the utility function
  
  – Money saves on leisure, which really belongs in the utility function
  
  – Therefore, money in the utility function is a useful short-cut to fully modeling the transactions technology whereby money saves on leisure

• Foreign money is missing due to assumption that domestic residents use only domestic money as a medium of exchange
2.2 Budget constraint

\[ B_{t+1} + \frac{M_t}{P_t} = (1 + r) B_t + \frac{M_{t-1}}{P_t} + Y_t - C_t - T_t \]

2.3 FO Conditions

- Solve the budget constraint for consumption and substitute into utility

- Maximize with respect to \( B_{s+1} \) and \( M_s \)
\[- B_{s+1} \]

\[
u_c \left( C_s, \frac{M_s}{P_s} \right) = \beta (1 + r) u_c \left( C_{s+1}, \frac{M_{s+1}}{P_{s+1}} \right)
\]

\[- M_s \]

\[
u_c \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_s} = u_{M/P} \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_s} + \beta u_c \left( C_{s+1}, \frac{M_{s+1}}{P_{s+1}} \right) \frac{1}{P_{s+1}}
\]

* marginal cost of money is the utility of the consumption the agent gives up to acquire money

* marginal benefit is the utility provided by money plus the utility of consumption the agent can obtain with the money next period
– Use Euler equation to rewrite FO condition on money by substituting out future marginal utility of consumption

$$u_c \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_s} = u_{M/P} \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_s} + u_c \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_{s+1}} \frac{1}{1 + r}$$

$$u_c \left( C_s, \frac{M_s}{P_s} \right) \left[ \frac{1}{P_s} - \frac{1}{P_{s+1}} \frac{1}{1 + r} \right] = u_{M/P} \left( C_s, \frac{M_s}{P_s} \right) \frac{1}{P_s}$$

$$u_c \left( C_s, \frac{M_s}{P_s} \right) \left[ 1 - \frac{P_s}{P_{s+1}} \frac{1}{1 + r} \right] = u_{M/P} \left( C_s, \frac{M_s}{P_s} \right)$$

* note that

$$1 + i_{s+1} = (1 + r) \frac{P_{s+1}}{P_s}$$
* substituting, the interest rate term becomes

\[ 1 - \frac{P_s}{P_{s+1}} \frac{1}{1 + r} = 1 - \frac{1}{1 + i_{s+1}} = \frac{i_{s+1}}{1 + i_{s+1}} \]

* substituting into FO condition on money

\[ \frac{i_{s+1}}{1 + i_{s+1}} = \frac{u_M/P}{u_C} \left( \frac{C_s}{M_s} \right) \left( \frac{M_s}{P_s} \right) \]

* marginal utility of money relative to marginal utility of consumption equals the opportunity cost of money

* we can view the equation as a money demand equation
Let utility be

\[ u(C, \frac{M}{P}) = \left[ C^{\gamma} \left( \frac{M}{P} \right)^{1-\gamma} \right]^{1-1/\sigma} \]

\[ \frac{i_{s+1}}{1 + i_{s+1}} = \frac{(1 - \gamma) C_s^{\gamma} \left( \frac{M_s}{P_s} \right)^{-\gamma}}{\gamma C_s^{\gamma-1} \left( \frac{M_s}{P_s} \right)^{1-\gamma}} \]

Solving for real money balances

\[ \frac{M_s}{P_s} = \frac{(1 - \gamma) C_s}{\gamma \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right]} = \frac{(1 - \gamma)}{\gamma} \left[ \frac{1 + i_{s+1}}{i_{s+1}} \right] C_s \]

money demand is decreasing in the nominal interest rate and increasing in consumption
2.4 Intertemporal Budget Constraint

- Agent’s flow budget constraint

\[ B_{t+1} - (1 + r) B_t = Y_t - C_t - T_t - \left( \frac{M_t - M_{t-1}}{P_t} \right) \]

- Take present discounted value and sum

\[
\sum_{s=t}^{\infty} \left[ B_{s+1} - (1 + r) B_s \right] \left( \frac{1}{1 + r} \right)^{s-t} \\
= \sum_{s=t}^{\infty} \left[ Y_s - C_s - T_s - \frac{M_s - M_{s-1}}{P_s} \right] \left( \frac{1}{1 + r} \right)^{s-t}
\]
– write out the money terms

\[-\frac{M_t}{P_t} + \frac{M_{t-1}}{P_t} - \left(\frac{1}{1+r}\right) \left(\frac{M_{t+1}}{P_{t+1}} - \frac{M_t}{P_{t+1}}\right) - \left(\frac{1}{1+r}\right)^2 \left(\frac{M_{t+2}}{P_{t+2}} - \frac{M_{t+1}}{P_{t+2}}\right) + \ldots\]
group coefficients on money in each period up to \( T \) periods

\[
\frac{M_{t-1}}{P_t} - \frac{M_t}{P_t} \left[ 1 - \frac{P_t}{P_{t+1}} \frac{1}{1 + r} \right] - \frac{M_{t+1}}{P_{t+1}} \left( \frac{1}{1 + r} \right) \left[ 1 - \frac{P_{t+1}}{P_{t+2}} \frac{1}{1 + r} \right] - \\
\ldots - \frac{M_{t+T}}{P_{t+T}} \left( \frac{1}{1 + r} \right)^T \\
= \frac{M_{t-1}}{P_t} - \frac{M_t}{P_t} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] - \frac{M_{t+1}}{P_{t+1}} \left( \frac{1}{1 + r} \right) \left[ \frac{i_{s+2}}{1 + i_{s+2}} \right] - \\
\ldots - \frac{M_{t+T}}{P_{t+T}} \left( \frac{1}{1 + r} \right)^T \\
= \frac{M_{t-1}}{P_t} - \sum_{s=t}^{T-1} \frac{M_s}{P_s} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] \left( \frac{1}{1 + r} \right)^{s-t} - \frac{M_{t+T}}{P_{t+T}} \left( \frac{1}{1 + r} \right)^T
\]
IBC becomes

$$\lim_{T \to \infty} \left( B_{t+T+1} + \frac{M_{t+T}}{P_{t+T}} \right) \left( \frac{1}{1 + r} \right)^T - (1 + r) B_t - \frac{M_{t-1}}{P_t}$$

$$= \sum_{s=t}^{\infty} \left[ Y_s - C_s - T_s - \frac{M_s}{P_s} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] \right] \left( \frac{1}{1 + r} \right)^{s-t}$$

- setting the limit term to zero and rearranging

$$\sum_{s=t}^{\infty} \left[ C_s + \frac{M_s}{P_s} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] \right] \left( \frac{1}{1 + r} \right)^{s-t}$$

$$= \sum_{s=t}^{\infty} [Y_s - T_s] \left( \frac{1}{1 + r} \right)^{s-t} + (1 + r) B_t + \frac{M_{t-1}}{P_t}$$

- present value of expenditures on consumption and real money balances must equal the present value of disposable income plus initial assets
2.5 Equilibrium path for prices and the exchange rate

2.5.1 Budget constraints and equilibrium consumption

- Government flow budget constraint
  \[ G_t = T_t + \frac{M_t - M_{t-1}}{P_t} \]
  - the government’s budget is always balanced with government spending equal to taxes plus seigniorage

- Agent flow budget constraint
  \[ B_{t+1} - (1 + r) B_t = Y_t - C_t - T_t - \left( \frac{M_t - M_{t-1}}{P_t} \right) \]
Combine to get country flow budget constraint

\[ B_{t+1} - (1 + r) B_t = Y_t - C_t - G_t \]

- Addition of money to the model has no effect on the country’s budget constraint because agents pay seigniorage revenues to the government.

Equilibrium when \( \beta (1 + r) = 1 \), yielding constant consumption \( (\bar{C}') \)

\[
\bar{C}' = r B_t + \frac{r}{1 + r} \sum_{s=t}^{\infty} [Y_s - G_s] \left( \frac{1}{1 + r} \right)^{s-t}
\]
2.5.2 Speculative Bubbles in the Price Level

- Simplifying assumptions
  - $G = 0$
  - $\beta (1 + r) = 1$
  - $C = \bar{C} = \bar{Y}$
  - money growth is constant

$$\frac{M_{t+1}}{M_t} = 1 + \mu$$
- utility is separable in consumption and real money balances

\[ u(C, \frac{M}{P}) = \log C + v \left( \frac{M}{P} \right) \quad \text{where } v \text{ is strictly concave} \]

- Money Euler equation

\[ u_c \left( \frac{1}{P_t} - \frac{1}{P_{t+1}(1+r)} \right) = \frac{1}{P_t} u_{M/P} \]

- substitute for marginal utilities and rearrange

\[ \frac{1}{P_t} - \frac{1}{P_{t+1}(1+r)} = \frac{1}{P_t} v' \left( \frac{M_t}{P_t} \right) \bar{C} \]
– multiply by $P_t$ and substitute $\beta = \frac{1}{1+r}$

$$1 - \beta \frac{P_t}{P_{t+1}} = v' \left( \frac{M_t}{P_t} \right) \bar{C}$$

– note that $\frac{M_{t+1}}{M_t} \frac{1}{1+\mu} = 1$, and multiply $\beta \frac{P_t}{P_{t+1}}$ by this

$$1 - \beta \frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t} \frac{1}{1+\mu} = v' \left( \frac{M_t}{P_t} \right) \bar{C}$$

– multiply by $\frac{M_t}{P_t}$ to get a dynamic equation in real money balances

$$\frac{M_t}{P_t} \left[ 1 - v' \left( \frac{M_t}{P_t} \right) \bar{C} \right] = \beta \frac{M_{t+1}}{P_{t+1}} \frac{1}{1+\mu}$$
– as a special case, let \( v \left( \frac{M}{P} \right) = \log \left( \frac{M}{P} \right) \)

\[
\frac{M_t}{P_t} - \bar{C} = \frac{M_{t+1}}{P_{t+1}} \frac{\beta}{1 + \mu}
\]

– solve for future real money balances as a function of current and draw phase diagram

\[
\frac{M_{t+1}}{P_{t+1}} = \frac{1 + \mu}{\beta} \left[ \frac{M_t}{P_t} - \bar{C} \right]
\]

* the dynamic equation is unstable since \( \frac{1 + \mu}{\beta} > 1 \)

* if current real money is low, due to a high nominal interest rate which implies high expectations of inflation, inflation rises reducing real money further
Can we rule out divergent paths, requiring price to jump to the equilibrium immediately?

- Derive the transversality condition using the Euler equation for real balances

\[ u_c \left( C_t, \frac{M_t}{P_t} \right) \frac{1}{P_t} = u_{M/P} \left( C_t, \frac{M_t}{P_t} \right) \frac{1}{P_t} + \beta u_c \left( C_{t+1}, \frac{M_{t+1}}{P_{t+1}} \right) \frac{1}{P_{t+1}} \]

- Let utility be

\[
\log C + v \left( \frac{M}{P} \right) \]

\[
\frac{1}{C_t P_t} = \frac{1}{P_t} v' \left( \frac{M_t}{P_t} \right) + \frac{\beta}{C_{t+1} P_{t+1}} \frac{1}{P_{t+1}}
\]
– Take one period forward

\[
\frac{1}{C_{t+1}} \frac{1}{P_{t+1}} = \frac{1}{P_{t+1}} v' \left( \frac{M_{t+1}}{P_{t+1}} \right) + \frac{\beta}{C_{t+2}} \frac{1}{P_{t+2}}
\]

– Substitute into the first equation

\[
\frac{1}{C_t} \frac{1}{P_t} = \frac{1}{P_t} v' \left( \frac{M_t}{P_t} \right) + \beta \left[ \frac{1}{P_{t+1}} v' \left( \frac{M_{t+1}}{P_{t+1}} \right) + \frac{\beta}{C_{t+2}} \frac{1}{P_{t+2}} \right]
\]

– Continue for \( T - 1 \) iterations

\[
\frac{1}{C_t} \frac{1}{P_t} = \sum_{s=t}^{t+T-1} \beta^{s-t} \frac{1}{P_s} v' \left( \frac{M_s}{P_s} \right) + \beta^T \frac{1}{C_{t+T}} \frac{1}{P_{t+T}}
\]

* marginal utility of 1 dollar’s worth of consumption equals the marginal utility of holding that dollar as money from periods \( t \) through \( T-1 \) plus the marginal utility of spending that dollar on consumption
in period \( t + T \)

* must hold along the optimal path

* additionally marginal utility of a dollar’s worth of consumption today must equal the marginal utility of holding that dollar as money forever

* therefore, there can be no gain from holding a dollar forever and never converting it to consumption

\[
\frac{1}{C_t} \frac{1}{P_t} = \sum_{s=t}^{\infty} \beta^{s-t} \frac{1}{P_s} u' \left( \frac{M_s}{P_s} \right)
\]

* together, the two conditions imply the transversality condition

\[
\lim_{T \to \infty} \beta^T \frac{1}{C_{t+T}} \frac{1}{P_{t+T}} = 0
\]
· transversality condition does not rule out rising prices

· it does rule out prices falling too fast
– Show that transversality condition rules out deflation for a constant money supply

* Let

\[
\frac{P_{t+1}}{P_t} = \xi < 1
\]

* Solving for price in the future as a function of past prices

\[
P_{t+1} = P_t \xi
\]

· Iterating forward

\[
P_{t+2} = P_{t+1} \xi = P_t \xi^2
\]

\[
P_{t+T} = P_t \xi^T
\]
* Rewrite the limit term, substituting for $P_{t+T}$

$$\lim_{T \to \infty} \frac{\beta^T}{\xi^T} \frac{1}{C_{t+T}} \frac{1}{P_t} = 0$$

- Recall $C_{t+T} = \tilde{C}$

- Transversality condition fails if

$$\frac{\beta}{\xi} \geq 1$$
* Must show that if prices are falling they will eventually fall too fast such that violate the transversality condition as $\frac{P_{t+1}}{P_t}$ gets too small

  · money Euler equation

  \[ \frac{1}{P_{t+1}} = \frac{1}{\beta P_t} - \frac{v'(\frac{M_t}{P_t}) \bar{C}}{\beta P_t} \]

  · take the limit as $P_t \to 0$

  · assume that there is an upper bound on the quantity of real balances that yield utility so that $v\left(\frac{M_t}{P_t}\right)$ is bounded from above

  · concavity of $v\left(\frac{M_t}{P_t}\right)$ implies

  \[ v\left(\frac{M}{P}\right) - v\left(\frac{M}{P_0}\right) > v'(\frac{M}{P}) \left[ \frac{M}{P} - \frac{M}{P_0} \right] \]
- upper bound on $v \left( \frac{M_t}{P_t} \right)$ implies that the left hand side is bounded, implying a bound on the right hand side

- limit of rhs is

$$\lim_{P \to 0} v' \left( \frac{M}{P} \right) \left[ \frac{M}{P} - \frac{M}{P_0} \right] = M \lim_{P \to 0} v' \left( \frac{M}{P} \right) \left[ \frac{1}{P} \right]$$

- therefore, in the limit, the money Euler equation implies

$$\frac{1}{P_{t+1}} = \frac{1}{\beta P_t} - H$$

- implying that eventually the inverse of price grows at rate $\frac{1}{\beta}$ as price gets so large that its inverse dwarfs fixed limit term $H$

- if prices are falling, eventually they must fall so fast that they violate the transversality condition
* If price starts out too low, then the marginal utility of consumption exceeds the present value of holding money forever making agents wish to reduce real money, instantly driving $P$ up

* rule out hyperdeflation with the transversality condition
speculative hyperinflation

- Money Euler equation with constant money growth
\[
\frac{M_{t+1}}{P_{t+1}} \frac{\beta}{1 + \mu} = \frac{M_t}{P_t} \left[ 1 - v' \left( \frac{M_t}{P_t} \right) \bar{C} \right]
\]

* unstable system

* if the marginal utility of money increases more slowly than money falls
\[
\lim_{M/P \to 0} \frac{M_t}{P_t} v' \left( \frac{M_t}{P_t} \right) = 0,
\]
then hyperinflation, which eliminates money is a possible equilibrium

- If price starts out too high, because interest is high with agents expecting rising prices, then prices rise, reducing real money balances
etc.

– in the penultimate period after which real money will be eliminated, the marginal utility of money equals the marginal utility of consumption for which it can be exchanged

\[ \frac{1}{C} = v' \left( \frac{M_t}{P_t} \right) \]

* to rule out hyperinflation, need the marginal utility of money to increase more rapidly than money falls

\[ \lim_{M/P \to 0} \frac{M_t}{P_t} v' \left( \frac{M_t}{P_t} \right) > 0 \]
* requires that money be essential

\[
\lim_{M/P \to 0} v \left( \frac{M_t}{P_t} \right) = -\infty
\]

* no finite quantity of consumption could compensate for giving up money
2.5.3 Fiscal Theory of the Price Level (FTPL)

- Assume domestic government issues nominal bonds and households hold nominal government bonds in addition to real private bonds

- Simplify by assuming PPP

\[ P_t = E_t P_t^* \]
• Household budget constraints

  – Household flow budget constraint, where $B_{t}^{gh*}$ is domestic household holdings of foreign bonds, and $E_t$ is the exchange rate

  \[
  B_{t+1} - (1 + r) B_t + \left( \frac{B_{t+1}^{gh} - (1 + r) B_{t}^{gh}}{P_t} \right) \\
  + E_t \left( \frac{B_{t+1}^{gh*} - (1 + r) B_{t}^{gh*}}{P_t} \right) \\
  = Y_t - C_t - T_t - \left( \frac{M_t - M_{t-1}}{P_t} \right)
  \]
Household intertemporal budget constraint

\[
\sum_{s=t}^{\infty} \left[ T_s + \left( \frac{M_s - M_{s-1}}{P_s} \right) \right] \left( \frac{1}{1 + r} \right)^{s-t} \\
= \sum_{s=t}^{\infty} (Y_s - C_s) \left( \frac{1}{1 + r} \right)^{s-t} + (1 + r) \left( B_t + \frac{B_t^{gh}}{P_t} + \frac{E_t B_t^{gh*}}{P_t} \right) \\
- \lim_{T \to \infty} \left( B_{t+T+1} + \frac{B_{t+T+1}^{gh}}{P_{t+T}} + \frac{E_t B_{t+T+1}^{gh*}}{P_{t+T}} \right) \left( \frac{1}{1 + r} \right)^T
\]
• Adding domestic and foreign household budget constraints with\( \frac{B_t^g}{P_t} \left( \frac{E_t B_t^{g*}}{P_t} \right) \) denoting the real outstanding stock of domestic (foreign) government bonds

\[
\sum_{s=t}^{\infty} \left[ T_s + \left( \frac{M_s - M_{s-1}}{P_s} \right) + T_s^* + \left( \frac{M_s^* - M_{s-1}^*}{P_s^*} \right) \right] \left( \frac{1}{1 + r} \right)^{s-t} \\
= \sum_{s=t}^{\infty} (Y_s + Y_s^* - C_s - C_s^*) \left( \frac{1}{1 + r} \right)^{s-t} + (1 + r) \left( \frac{B_t^g}{P_t} + \frac{E_t B_t^{g*}}{P_t} \right) \\
- \lim_{T \to \infty} \left( \frac{B_{t+T+1}^g}{P_{t+T}} + \frac{E_t B_{t+T+1}^{g*}}{P_{t+T}} \right) \left( \frac{1}{1 + r} \right)^T
\]

• Goods market equilibrium (global resource constraint which must hold in equilibrium)

\[ Y_s + Y_s^* - C_s - C_s^* = G_s + G_s^* \]
Substituting into global agent budget constraint yields the sum of government intertemporal budget constraints

\[
\sum_{s=t}^{\infty} \left[ T_s + \left( \frac{M_s - M_{s-1}}{P_s} \right) + T_s^* + \left( \frac{M_s^* - M_{s-1}^*}{P_s^*} \right) \right] \left( \frac{1}{1 + r} \right)^{s-t}
\]

\[
= \sum_{s=t}^{\infty} (G_s + G_s^*)(\frac{1}{1 + r})^{s-t} + (1 + r) \left( \frac{B_{t}^g}{P_t} + \frac{E_t B_{t}^{g*}}{P_t} \right)
\]

\[
- \lim_{T \to \infty} \left( \frac{B_{t+T}^g}{P_{t+T}} + \frac{E_t B_{t+T}^{g*}}{P_{t+T}} \right) \left( \frac{1}{1 + r} \right)^T
\]
• Limit term in equilibrium

- * domestic NPG together with optimal behavior implies

\[ \lim_{T \to \infty} (B_{t+T+1}) \left( \frac{1}{1 + r} \right)^T = \lim_{T \to \infty} \left[ \frac{B_{t+T+1}^{gh}}{P_{t+T}} + \frac{E_t B_{t+T+1}^{gh*}}{P_{t+T}} \right] \left( \frac{1}{1 + r} \right)^T \]

- foreign NPG together with optimal behavior implies

\[ \lim_{T \to \infty} (B_{t+T+1}) \left( \frac{1}{1 + r} \right)^T = \lim_{T \to \infty} \left[ \frac{B_{t+T+1}^{gf}}{P_{t+T}} + \frac{E_t B_{t+T+1}^{gf*}}{P_{t+T}} \right] \left( \frac{1}{1 + r} \right)^T \]

- adding implies NPG for governments together

\[ 0 = \lim_{T \to \infty} \left( \frac{B_{t+T+1}^g}{P_{t+T}} + \frac{E_t B_{t+T+1}^{g*}}{P_{t+T}} \right) \left( \frac{1}{1 + r} \right)^T \]
* households can satisfy their NPG condition if one country’s government debt grows indefinitely as long as the other country’s government has offsetting growth in assets

* equivalently one government finances the other

* There are an infinite number of prices and exchange rates consistent with zero limit term on world government debt
• Assumption necessary for equilibrium behavior to require a zero limit term for each government’s debt separately

  – No government wants to tax its citizens to accumulate debt of another indefinitely

  – NPG constraint across governments

  – If no government allows its assets to grow at a rate faster than the interest rate, then no government’s debt can grow faster than the interest rate

• Government intertemporal budget constraint with limit term imposed

\[
\frac{(1 + r) B_t^g}{P_t} = \sum_{s=t}^{\infty} \left[ T_s + \left( \frac{M_s - M_{s-1}}{P_s} \right) - G_s \right] \left( \frac{1}{1 + r} \right)^{s-t}
\]
– Government budget constraint is an equilibrium condition imposed by the behavior of households and other governments in equilibrium.
- Government could decide to cut taxes forever without adjusting any other element in budget constraint

* households would perceive an increase in present-value resources and attempt an increase in consumption

* with output at full employment, price rises reducing real value of government bonds (wealth) until households want no additional consumption

* the price level is determined by the expected present value of future primary government surpluses and initial government debt

\[
\frac{(1 + r) B_t^g}{P_t} = \sum_{s=t}^{\infty} [S_s] \left( \frac{1}{1 + r} \right)^{s-t}
\]

* initial value for price is determined
* money supply must adjust to assure money market equilibrium at that price

  - if money supply fails to adjust, hyperinflation is possible

  - sometimes no equilibrium

  - monetary policy must be passive, when fiscal policy is active
– Alternative expression

\[
\sum_{s=t}^{\infty} \left[ \frac{M_s - M_{s-1}}{P_s} \right] \left( \frac{1}{1 + r} \right)^{s-t} = \frac{M_{t-1}}{P_t} - \sum_{s=t}^{\infty} \frac{M_s}{P_s} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] \left( \frac{1}{1 + r} \right)^{s-t} - \lim_{T \to \infty} \frac{M_{t+T}}{P_{t+T}} \left( \frac{1}{1 + r} \right)^T
\]

– Transversality condition on the household assures limit on real money term is zero

– Substituting

\[
\frac{(1 + r) B_t^g + M_{t-1}}{P_t} = \sum_{s=t}^{\infty} \left[ T_s + \frac{M_s}{P_s} \left[ \frac{i_{s+1}}{1 + i_{s+1}} \right] - G_s \right] \left( \frac{1}{1 + r} \right)^{s-t}
\]
• Government behavior determines whether the FTPL holds

  – Is the price level unique?

  – Consider price increase which reduces outstanding real debt

  – If government adjusts primary surpluses to validate the price increase, then any price is possible and FTPL does not hold (passive fiscal policy)

  – If government does not adjust, then new price is not an equilibrium, and price must return to assure intertemporal government budget balance (active fiscal policy)
3 Cash in Advance Model

3.1 Optimization problem

- maximize utility
  \[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(c_s) \]

- subject to the flow budget constraint
  \[ B_{t+1} + \frac{M_t}{P_t} = (1 + r) B_t + \frac{M_{t-1}}{P_t} + Y_t - C_t - T_t \]
• and a cash in advance constraint

\[ M_{t-1} \geq P_t C_t \]

– if the nominal interest rate is positive and there is no uncertainty, then the cash in advance constraint will hold with equality
• substituting the cash in advance constraint with equality into the budget constraint

\[ B_{t+1} + \frac{P_{t+1}C_{t+1}}{P_t} = (1 + r)B_t + C_t + Y_t - C_t - T_t \]

– rearranging

\[ B_{t+1} = (1 + r)B_t + Y_t - T_t - \frac{P_{t+1}C_{t+1}}{P_t} \]

• solve flow budget constraint and cash in advance constraint each for consumption

\[ C_{t+1} = \frac{P_t}{P_{t+1}} \left[ (1 + r)B_t + Y_t - T_t - B_{t+1} \right] \]

\[ C_t = P_tM_{t-1} \]
– in general

\[ C_s = \frac{P_{s-1}}{P_s} [(1 + r) B_{s-1} + Y_{s-1} - T_{s-1} - B_s] \quad s > t \]

with \( C_t \) pre-determined
• Substitute $C_s$ into the utility function

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u\left( \frac{P_{s-1}}{P_s} \left[ (1 + r) B_{s-1} + Y_{s-1} - T_{s-1} - B_s \right] \right)$$

• FO condition for $B_s$ for $s > t$

$$u'(C_s) \frac{P_{s-1}}{P_s} = \beta (1 + r) u'(C_{s+1}) \frac{P_s}{P_{s+1}}$$

- Divide both sides by $(1 + r)$

$$\frac{u'(C_s)}{(1 + r) \frac{P_s}{P_{s-1}}} = \beta \frac{1 + r}{(1 + r) \frac{P_{s+1}}{P_s}} u'(C_{s+1})$$
– Equivalently

\[
\frac{u'(C_s)}{(1 + i_s)} = \frac{\beta (1 + r) u'(C_{s+1})}{(1 + i_{s+1})}
\]

– If inflation is changing, it provides a distortion in Euler equation convincing agents to substitute away from consumption in high-inflation periods

– If inflation is not changing, then cash in advance introduces no distortions
3.2 Money Demand

- No opportunity cost variable

$$\frac{M_{t-1}}{P_t} = C_t$$

- Get opportunity cost variable (interest rate)
  - Add credit goods which do not have to be purchased with money
    * increase in interest rate raises the relative cost of cash goods reducing the demand for them, thereby reducing money demand
  - Add uncertainty about future consumption
Agents hold more money than necessary to purchase expected consumption to have some precautionary balances.

Increase in the interest rate raises the cost of these precautionary balances, reducing money demand.
4 Fixing the Exchange Rate

- Solution for the exchange rate from monetary model

\[ e_t = m_t - m_t^* + \eta (E_t e_{t+1} - e_t) - \phi (y_t - y_t^*) \]

- The government wants to permanently fix the exchange rate at

\[ e_t = \bar{e} \]

- Solve for time path of the money supply that will fix the exchange rate

\[ m_t = \bar{e} + m_t^* + \phi (y_t - y_t^*) \]

* money supply must move directly with foreign money supply and with changes in relative outputs
• Fixed exchange rate policy requires a country to give up independent monetary policy

• Fixing the exchange rate in practice
  – Central bank does not actually work with the equation above
  – It states an exchange rate and promises to buy or sell foreign currency in any quantity at that rate
  – Must have a stock of foreign currency to implement the policy, foreign exchange reserves
• Other ways of implementing a fixed exchange rate
  
  – Obstfeld and Rogoff add government spending and show that government spending can be manipulated to fix the exchange rate
  
  – Would never work in practice because government spending cannot react sufficiently quickly

• FTPL implies that fiscal policy must be passive
  
  – Present value of future surpluses is fixed at the real value of outstanding government debt at the fixed exchange rate
5 First Generation Exchange Rate Crises

5.1 Assumptions

- Continuous time model with lower-case letters the logarithms of upper case letters

- Money demand

\[ m_t - e_t = -\eta \dot{e}_t \]

where all other variables entering money demand have been normalized at zero
• Fixed exchange rate regime is $\dot{e}_t = 0$, requiring

\[ m_t = \bar{m} = \bar{e} \]

• Money supply is backed by home government bonds and foreign exchange reserves (foreign government bonds)

\[ M_t = B_{H,t} + \bar{E}B_{F,t} \]
• Combined monetary and fiscal policy

  – government spending and tax policy is fixed at levels which imply a deficit and the need to issue government bonds

  – monetary authority is financing a government deficit by buying government bonds

\[
\frac{\dot{B}_{H,t}}{B_{H,t}} = \dot{b}_{H,t} = \mu > 0
\]

  – since the money supply cannot increase, the government is losing foreign exchange reserves

\[
\dot{M}_t = \dot{B}_{H,t} + \bar{E}\dot{B}_{F,t} = 0
\]

\[
\dot{B}_{H,t} = -\bar{E}\dot{B}_{F,t}
\]
– central bank effectively buys government bonds using its foreign exchange reserves

• There is a lower bound on foreign exchange reserves of zero

– Therefore, policy is not sustainable

– Given the assumption that government tax and spending policy cannot change, then eventually after reserves are eliminated

\[ M_t = \dot{B}_{H,t} \]

and the fixed exchange rate must fail
5.2 Post-Collapse Equilibrium

- Once the central bank has run out of reserves, the money supply must grow at rate \( \mu \)

\[ \dot{m}_t = \mu \]

- Therefore, the exchange rate must rise at rate \( \mu \)

\[ \dot{e}_t = \mu \]
• On the date on which the monetary authority runs out of reserves, money demand falls discretely by $\eta\mu$
  
  – Requires a reduction in real money supply $m_t - e_t$

  – the exchange rate cannot jump along a perfect foresight path where its expected rate of change is zero

  – therefore, on the collapse day, the money supply must jump discretely downwards

• Speculative attack whereby agents buy monetary authority’s remaining foreign exchange reserves ($B_F$) with domestic money, thereby reducing the money supply
5.3 Collapse Timing

- Define a shadow exchange rate as the rate which would prevail if all foreign exchange reserves were exhausted

\[ \tilde{e}_t = b_{H,t} + \eta \mu \]

where post-collapse

\[ m_t = b_{H,t} \quad \dot{m}_t = \dot{b}_{H,t} = \dot{e}_t = \mu \]

- Since \( b_{H,t} \) is rising, the shadow exchange rate is rising

- A speculative attack will occur on the date at which

\[ \tilde{e}_t = e_t \]
– Agents will use domestic money to buy the remaining foreign exchange reserves, creating a discrete fall in the money supply equal to the remaining foreign exchange reserves
• Solve for time of speculative attack

  – Monetary authority’s stock of government bonds is rising

\[ b_{H,t} = b_{H,0} + \mu t \]

  – Shadow foreign exchange rate is rising

\[ \tilde{e}_t = b_{H,t} + \eta \mu = b_{H,0} + \mu t + \eta \mu \]

  – Collapse date \( T \) is the date on which the shadow exchange rate equals the fixed rate

\[ \bar{e} = b_{H,0} + \mu T + \eta \mu \]

\[ T = \frac{\bar{e} - b_{H,0} - \eta \mu}{\mu} \]
– Recognize that

\[ \tilde{e} = \log \left( B_{H,t} + \bar{E} B_{F,t} \right) \]

\[ T = \frac{\log \left( B_{H,t} + \bar{E} B_{F,t} \right) - b_{H,0} - \eta \mu}{\mu} \]

* the time to collapse is increasing in foreign exchange reserves

* and decreasing in the post-collapse rate of growth of the money supply
5.4 Major Results

- Large speculative attack is fully predictable and is independent of external precipitating events

- The collapse is due to inconsistent monetary and fiscal policy
  
  - Fiscal policy requires financing from monetary authority
  
  - Monetary authority wants zero inflation and therefore provides no seigniorage as long as it has reserves
  
  - Once reserves are gone, monetary authority must provide the seigniorage and allow the money supply to grow
• Collapse of the fixed exchange rate occurs with no change in the value of the exchange rate
5.5 Extensions of the Model

• Discrete time

  – crisis occurs at an integer $T$ where

    $$\tilde{e} \geq \bar{e}$$

  – requires an earlier reserve-reducing speculative attack to get $M$ down to that required given the expectation that $e$ will rise on the speculative attack date

• Uncertainty about rate of growth of money supply and discrete time

  – same results as above with addition that timing becomes uncertain
6 Second Generation Exchange Rate Crisis (Obstfeld 1986)

6.1 Assumptions about government policy

- If there is no attack on the fixed exchange rate system, then domestic credit grows at rate $\mu_0$

- If there is an attack, then domestic credit grows at rate $\mu_1$ where

  \[ \mu_1 > \mu_0 = 0 \]
6.2 Equilibrium

• Shadow exchange rate for each rate of growth of domestic credit

• There is a range for the value of domestic credit for which there are multiple equilibria
  
  – All agents believe that there will be no attack
    
    ★ Equilibrium is a fixed exchange rate

    ★ If a single agent were to deviate, with all other agents maintaining belief in fixed rate, he would expect price of the currency to fall to the shadow rate associated with $\mu_0 = 0$, and he would lose

  – All agents believe there will be an attack
* All attack, buying at fixed rate and selling at higher shadow rate, making profits

– The fixed rate is viable indefinitely as long as all believe it is viable

• Role for purely speculative behavior in creating an exchange rate crisis